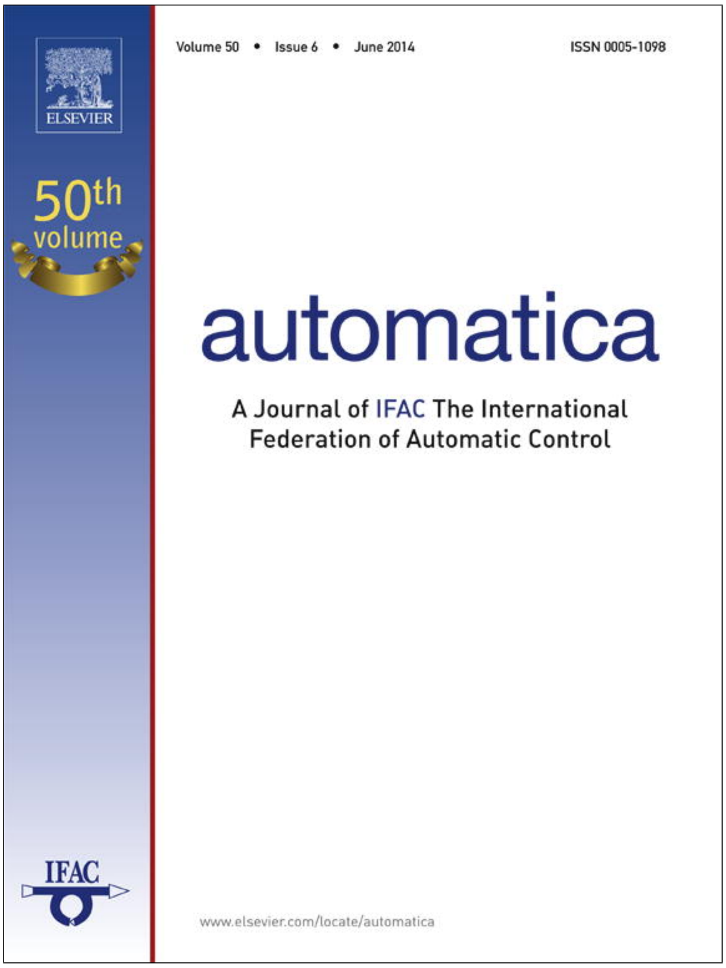
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A study of disturbance observers with unknown relative degree of the plant[☆]

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ABSTRACT

Robust stability of the disturbance observer (DOB) control system is studied when the relative degree of the plant is not the same as that of the nominal model. The study reveals that the closed-loop system can easily become unstable with sufficiently fast Q-filter when the relative degree of the plant is not known. In a few cases of unknown relative degree, however, robust stability can be obtained, and we present a design guideline of the nominal model, as well as the Q-filter, for that purpose. Moreover, a universal design of DOB is given for a plant whose relative degree is uncertain but less than or equal to four.

1. Introduction

The disturbance observer (DOB) based controller has been widely used among control engineers since it has a powerful ability of uncertainty compensation and disturbance attenuation. (See, e.g., Kempf & Kobayashi, 1999, Kobayashi, Katsura, & Ohnishi, 2007, Lee & Tomizuka, 1996, Wang & Tomizuka, 2004 and Yi, Chang, & Shen, 2009 and references therein.) The standard DOB control system is illustrated in Fig. 1. In the figure, $P(s)$ and $P_n(s)$ represent the transfer functions of the uncertain plant and its nominal model, and signals d and r represent the input disturbance and the reference, respectively. It is assumed that $P(s) \in \mathcal{P}$ where $\mathcal{P}$ is a known set of uncertain plants. The controller $C(s)$ is designed for the nominal model $P_n(s)$. The Q-filter $Q(s)$ is a *stable* low-pass filter, which usually has the form (Choi, Yang, Chung, Kim, & Suh, 2003; Lee & Tomizuka, 1996; Shim & Jo, 2009) of

$$Q(s) = \frac{b_k(\tau s)^k + b_{k-1}(\tau s)^{k-1} + \cdots + b_0}{(\tau s)^l + a_{l-1}(\tau s)^{l-1} + \cdots + a_1(\tau s) + a_0} \tag{1}$$

where $\tau > 0$ is the filter time constant, k and l are nonnegative integers with $b_k \neq 0$. We assume $a_0 = b_0$ and $l - k \geq \mathrm{r.deg}(P_n)$, where $\mathrm{r.deg}(P_n)$ stands for relative degree of P_n.

The output y is represented as $y(s) = T_{yr}(s)r(s) + T_{yd}(s)d(s)$ where T_{yr} is the transfer function from r to y and so on. For sufficiently small $\tau > 0$, it can be shown that $T_{yr}(j\omega) \approx P_nC/(1 + P_nC)(j\omega)$ and $T_{yd}(j\omega) \approx 0$ on a finite frequency range, which implies the recovery of the nominal closed-loop steady-state performance. (See, e.g., Shim & Jo, 2009 and Shim & Joo, 2007 for more details.) This property holds only when all the transfer functions are stable. Therefore, the question of interest is the robust stability of the closed-loop system in Fig. 1 under the variation of $P(s) \in \mathcal{P}$, which depends on the selection of $Q(s)$ and $P_n(s)$. This question has been studied under the perspective of small-gain theorem in Choi et al. (2003), Kim and Chung (2003), Kong and Tomizuka (2013) and Schrijver and van Dijk (2002), where only a sufficient stability condition is presented that is conservative in nature. On the other hand, some necessary and sufficient condition for robust stability is presented in Shim and Jo (2009) and Shim and Joo (2007) under the assumption that the time constant of the Q-filter is sufficiently small, which has played a key role in extending to nonlinear systems (Back & Shim, 2008) and embedding an internal

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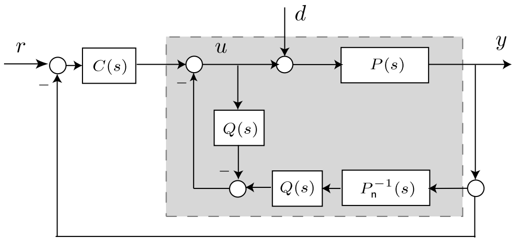

Fig. 1. Structure of the DOB control system. The shaded region represents the real plant $P(s)$ augmented with the DOB.

model into the DOB structure (Park, Joo, Shim, & Back, 2012). However, the study of Back and Shim (2008), Park et al. (2012), Shim and Jo (2009) and Shim and Joo (2007) assumes that the relative degree of plant is the same as that of nominal model.

In this paper,[2] we study the robust stability of the DOB-based control system (Fig. 1) when the relative degree of plant is not exactly known and so it happens to be different from that of nominal model. This case often occurs in real world control applications. For instance, $\mathrm{r.deg}(P) > \mathrm{r.deg}(P_n)$ when the actuator dynamics is ignored, or when there is unmodeled dynamics for the plant. Inspired by the fact that the characteristic equation for stability is of the form that appears in the 'higher-order root locus technique' (Hahn, 1981), a condition for robust stability is derived by utilizing the Newton diagram. The derived condition reveals a few facts such as: (1) if $\mathrm{r.deg}(P) = \mathrm{r.deg}(P_n) + 1$, robust stability can be achieved by an appropriate design of P_n as well as Q. (2) If $1 \le \mathrm{r.deg}(P) \le 2$, then robust stability is always achievable. (3) If $\mathrm{r.deg}(P) \ge \mathrm{r.deg}(P_n) + 2$ or $\mathrm{r.deg}(P_n) > \mathrm{r.deg}(P) > 2$, then robust stabilization is not possible with sufficiently small τ no matter how P_n, C, and Q are selected. A universal design of DOB is also discussed for the special case where $\mathrm{r.deg}(P)$ is unknown but $1 \le \mathrm{r.deg}(P) \le 4$.

Notation. Let $D(s)$ be a polynomial with real coefficients expressed as $D(s) = d_n s^n + d_{n-1}s^{n-1} + \cdots + d_1 s + d_0$. The polynomial $D(s)$ is said to be of *degree n* if $d_n \neq 0$, which will be denoted by $\deg(D) = n$. For a transfer function $G(s) = N(s)/D(s)$ (it is assumed that $N(s)$ and $D(s)$ are coprime polynomials), the degree and the relative degree of $G(s)$ are defined as $\deg(D)$ and $\deg(D) - \deg(N)$, respectively, and the latter will be denoted by $\mathrm{r.deg}(G)$. The high-frequency gain of $G(s)$ is defined as $\lim_{s\to\infty} s^{\mathrm{r.deg}(G)}G(s)$ and denoted by $\kappa(G)$. Finally, LHP (RHP, respectively) stands for the *open* left (right, respectively) half plane.

2. Robust stability

We assume that $P(s)$ and $P_n(s)$ are strictly proper while $C(s)$ is proper. Let P, P_n, C, and Q in Fig. 1 be represented by the ratios of coprime polynomials, that is, $P(s) = N(s)/D(s)$, $P_n(s) = N_n(s)/D_n(s)$, $C(s) = N_c(s)/D_c(s)$, and $Q(s) = N_Q(s;\tau)/D_Q(s;\tau)$ (in which, the dependence of N_Q and D_Q on τ is explicitly indicated). Moreover, we assume that there is no unstable pole-zero cancellation in $P_n(s)C(s)$ and in $P_n^{-1}(s)Q(s)$. Then, it can be shown that, for given $\tau > 0$, the closed-loop system is internally stable if and only if the characteristic polynomial

$$\delta(s;\tau) := (DD_c + NN_c)N_n D_Q + N_Q D_c(ND_n - N_n D)$$

is Hurwitz. Define

$$\begin{aligned} p_\alpha(s) &:= N(N_c N_n + D_c D_n) \\ p_\beta(s) &:= N_n(N_c N + D_c D) \end{aligned} \tag{2}$$

and let $m_\alpha := \deg(ND_cD_n)$, $m_\beta := \deg(N_nD_cD)$, and α_i, β_i be such that

$$p_\alpha(s) = \alpha_{m_\alpha}s^{m_\alpha} + \alpha_{m_\alpha-1}s^{m_\alpha-1} + \cdots + \alpha_0$$
$$p_\beta(s) = \beta_{m_\beta}s^{m_\beta} + \beta_{m_\beta-1}s^{m_\beta-1} + \cdots + \beta_0.$$

It should be kept in mind that $m_\beta - m_\alpha = \mathrm{r.deg}(P) - \mathrm{r.deg}(P_n)$, and that $\beta_{m_\beta}/\alpha_{m_\alpha} = \kappa(P_n)/\kappa(P)$. Let $\bar{k}$ ($\le k$) be such that $a_0 = b_0, \ldots, a_{\bar{k}} = b_{\bar{k}}$, and $a_{\bar{k}+1} \neq b_{\bar{k}+1}$ in $Q(s)$. Then, it follows that (with $a_l = 1$ for convenience)

$$\begin{aligned} \delta(s;\tau) &= p_\beta(s)D_Q(s;\tau) + (p_\alpha(s) - p_\beta(s))N_Q(s;\tau) \\ &= p_\beta(s)\sum_{i=0}^{l} a_i(\tau s)^i + (p_\alpha(s) - p_\beta(s))\sum_{i=0}^{k} b_i(\tau s)^i \\ &= \sum_{i=0}^{\bar{k}}(\tau s)^i a_i p_\alpha(s) + \sum_{i=\bar{k}+1}^{k}(\tau s)^i\big(a_i p_\beta(s) \\ &\quad + b_i(p_\alpha(s) - p_\beta(s))\big) + \sum_{i=k+1}^{l}(\tau s)^i a_i p_\beta(s). \end{aligned} \tag{3}$$

Note that $\deg(\delta(s;\tau)) = l + m_\beta$ if $\tau > 0$, and the locations of $l + m_\beta$ roots, when τ is sufficiently small, are of interest. Since $\delta(s;0) = a_0 p_\alpha(s)$ and $\deg(\delta(s;0)) = m_\alpha$, it is clear that m_α roots out of $l+m_\beta$ roots of $\delta(s;\tau)$ converge to the roots of $p_\alpha(s)$ as $\tau \to 0$, while the remaining $l + m_\beta - m_\alpha$ roots tend to infinity (see Shim & Jo, 2009 for more rigorous arguments).

Here we recall the result of Shim and Jo (2009), with the set $\mathcal{P}$ being a collection of transfer functions whose coefficients belong to certain (known) bounded intervals.

Proposition 1 (*Shim & Jo, 2009*). *Suppose that* $\mathrm{r.deg}(P) = \mathrm{r.deg}(P_n)$ *for all* $P(s) \in \mathcal{P}$. *Then, there exists a constant* $\tau^* > 0$ *such that, for all* $0 < \tau \le \tau^*$, *the closed-loop system is robustly stable if all the following conditions hold:*

(H1) *all* $P(s) \in \mathcal{P}$ *are of minimum phase,*
(H2) *the transfer function* $P_nC/(1 + P_nC)$ *is stable,*
(H3) *the polynomial*

$$p_f(s) := D_Q(s;1) + \left(\lim_{s\to\infty}\frac{P(s)}{P_n(s)} - 1\right)N_Q(s;1)$$

is Hurwitz for all $P(s) \in \mathcal{P}$.

On the contrary, for given $P \in \mathcal{P}$, *there is* $\tau^* > 0$ *such that, for all* $0 < \tau \le \tau^*$, *the closed-loop system is unstable if* $P_nC/(1 + P_nC)$ *has some poles in the RHP, or some zeros of* $P(s)$ *or some roots of* $p_f(s) = 0$ *are located in the RHP.*

Remark 2. It is observed that the conditions (H1) and (H2) are equivalent to $p_\alpha(s)$ being Hurwitz (see (2)), so that m_α roots of $\delta(s;\tau)$ have negative real parts for sufficiently small τ. On the other hand, the condition (H3) constrains the other $l+m_\beta-m_\alpha = l$ (since $m_\beta = m_\alpha$ if $\mathrm{r.deg}(P) = \mathrm{r.deg}(P_n)$) roots to remain in the LHP.

Note that Proposition 1 is not conclusive when any one of conditions is marginal (e.g., if some roots of $p_f(s)$ are located on the imaginary axis), by which the condition is 'almost' necessary and sufficient. It is inconclusive particularly when $\mathrm{r.deg}(P) > \mathrm{r.deg}(P_n)$ because $\lim_{s\to\infty} P(s)/P_n(s) = 0$ so that $p_f(s)$ has at least one root at the origin of complex plane (recall that $a_0 = b_0$). The polynomial $p_f(s)$ is not even defined when $\mathrm{r.deg}(P) < \mathrm{r.deg}(P_n)$.

[2] Preliminary versions of this paper have been presented at *Int. Conf. on Control, Automation and Systems, 2011*, where the Q-filter is just a first order system and the relative degree of P_n is limited to one, and at *51st IEEE Conf. on Dec. and Control, 2012*, where the case $\mathrm{r.deg}(P) < \mathrm{r.deg}(P_n)$ is not considered.

When $\mathrm{r.deg}(P) \neq \mathrm{r.deg}(P_n)$, the $l + m_\beta - m_\alpha$ roots of $\delta(s;\tau)$, that go to infinity as $\tau \to 0$, are of particular interest. To see their behavior conveniently, we want to make them go to zero as $\tau \to 0$. This is done by defining $\bar{\delta}(s;\tau) := s^{l+m_\beta}\delta(1/s;\tau)$. Then,

$$\bar{\delta}(s;\tau) = \bar{q}_0(s) + \tau\bar{q}_1(s) + \cdots + \tau^l \bar{q}_l(s), \tag{4}$$

$$\bar{q}_i(s) = \begin{cases} a_i\left(\alpha_{m_\alpha} s^{l+m_\beta-m_\alpha-i} + \cdots + \alpha_0 s^{l+m_\beta-i}\right) \\ \quad i = 0, 1, \ldots, \bar{k} \\ (a_i - b_i)(\beta_{m_\beta} s^{l-i} + \cdots + \beta_0 s^{l+m_\beta-i}) \\ \quad + b_i(\alpha_{m_\alpha} s^{l+m_\beta-m_\alpha-i} + \cdots + \alpha_0 s^{l+m_\beta-i}) \\ \quad i = \bar{k}+1, \bar{k}+2, \ldots, k \\ a_i(\beta_{m_\beta} s^{l-i} + \cdots + \beta_0 s^{l+m_\beta-i}) \\ \quad i = k+1, k+2, \ldots, l. \end{cases}$$

Since $\mathrm{Re}(s) < 0$ if and only if $\mathrm{Re}(1/s) < 0$ for a complex variable s, stability analysis using $\bar{\delta}$, instead of δ, is justified. As $\tau \to 0$, $l + m_\beta - m_\alpha$ roots of $\bar{\delta}$ are converging to zero whereas the remaining roots converge to m_α nontrivial roots of $\bar{q}_0(s)$. From now on, the former are called as vanishing roots while the latter as non-vanishing roots. Since $\bar{q}_0(s)/s^{l+m_\beta-m_\alpha} = a_0(\alpha_{m_\alpha} + \cdots + \alpha_0 s^{m_\alpha})$, the non-vanishing m_α roots have negative real parts if and only if $p_\alpha(s)$ is Hurwitz. Hence, paying attention to the vanishing roots, we can obtain the following Theorems 3 and 4 (for the case $\mathrm{r.deg}(P) > \mathrm{r.deg}(P_n)$) and Theorems 5 and 6 (for $\mathrm{r.deg}(P) < \mathrm{r.deg}(P_n)$), whose proofs are given in Section 4.

Theorem 3. *Suppose that* $\mathrm{r.deg}(P) = \mathrm{r.deg}(P_n) + 1$, $\forall P \in \mathcal{P}$. *Then, there exists* τ^* *such that, for all* $0 < \tau \leq \tau^*$, *the closed-loop system is robustly stable if both* (H1) *and* (H2) *hold and the following three conditions hold:*

(i) $\pi(s) := s^{l-1} + \cdots + a_{k+1}s^k + (a_k - b_k)s^{k-1} + \cdots + (a_1 - b_1)$ *is Hurwitz,*
(ii) $\kappa(P)$ *and* $\kappa(P_n)$ *have the same sign (i.e.,* $\alpha_{m_\alpha}\beta_{m_\beta} > 0$*),* $\forall P \in \mathcal{P}$,
(iii) $\sigma_+ := \frac{\alpha_{m_\alpha-1}}{\alpha_{m_\alpha}} - \frac{\beta_{m_\beta-1}}{\beta_{m_\beta}} + \frac{\alpha_{m_\alpha}}{\beta_{m_\beta}}\frac{a_0}{a_1-b_1}\left(\frac{a_2-b_2}{a_1-b_1} - \frac{b_1}{a_0}\right) < 0$,[3] $\forall P \in \mathcal{P}$.

As will be discussed in Section 3, the condition (iii) reveals that P_n should be carefully chosen for the robust stability, which is not the case in Shim and Jo (2009) where $\mathrm{r.deg}(P) = \mathrm{r.deg}(P_n)$. The conditions of Theorem 3 are almost necessary and sufficient in the following sense.

Theorem 4. *For given* $P \in \mathcal{P}$ *with* $\mathrm{r.deg}(P) > \mathrm{r.deg}(P_n)$*, the closed-loop system is unstable for sufficiently small* τ *if one of the following holds:* (a) $\mathrm{r.deg}(P) \geq \mathrm{r.deg}(P_n) + 2$, (b) P *has at least one RHP zero (violation of* (H1)*),* (c) $P_nC/(1+P_nC)$ *has at least one RHP pole (violation of* (H2)*),* (d) $\pi(s)$ *has at least one RHP root,* (e) $\alpha_{m_\alpha}\beta_{m_\beta} < 0$, (f) $\sigma_+ > 0$, (g) $\bar{k} > 0$.

Theorem 5. *Suppose that* $\mathrm{r.deg}(P) < \mathrm{r.deg}(P_n)$, $\forall P \in \mathcal{P}$. *Then, there exists* τ^* *such that, for all* $0 < \tau \leq \tau^*$, *the closed-loop system is robustly stable if both* (H1) *and* (H2) *hold, and, for all* $P \in \mathcal{P}$,

(i) $\mathrm{r.deg}(Q) \leq \mathrm{r.deg}(P_n) - \mathrm{r.deg}(P) + 2$,
(ii) $N_Q(s;1)$ *is Hurwitz (or a constant),*
(iii) $\kappa(P)$ *and* $\kappa(P_n)$ *have the same sign (i.e.,* $\alpha_{m_\alpha}\beta_{m_\beta} > 0$*) if* $\mathrm{r.deg}(Q) \geq \mathrm{r.deg}(P_n) - \mathrm{r.deg}(P) + 1$,
(iv) $\sigma_- := b_{k-1} - a_{l-1}b_k < 0$ *if* $\mathrm{r.deg}(Q) = \mathrm{r.deg}(P_n) - \mathrm{r.deg}(P) + 2$ *and* $k \geq 1$.

Since $\mathrm{r.deg}(Q) \geq \mathrm{r.deg}(P_n)$, the condition (i) of Theorem 5 imposes the restriction that $\mathrm{r.deg}(P) \leq 2$.

[3] $b_1 = 0$ if b_1 is not present in (1), and so on.

Theorem 6. *For given* $P \in \mathcal{P}$ *with* $\mathrm{r.deg}(P) < \mathrm{r.deg}(P_n)$*, the closed-loop system is unstable for sufficiently small* τ *if one of the following holds:* (a) $\mathrm{r.deg}(Q) \geq \mathrm{r.deg}(P_n) - \mathrm{r.deg}(P) + 3$, (b) P *has at least one RHP zero,* (c) $P_nC/(1+P_nC)$ *has at least one RHP pole,* (d) $N_Q(s;1)$ *has at least one RHP root,* (e) $\alpha_{m_\alpha}\beta_{m_\beta} < 0$ *while* $\mathrm{r.deg}(Q) \geq \mathrm{r.deg}(P_n) - \mathrm{r.deg}(P) + 1$, (f) $\sigma_- > 0$ *while* $\mathrm{r.deg}(Q) = \mathrm{r.deg}(P_n) - \mathrm{r.deg}(P) + 2$ *and* $k \geq 1$.

According to Theorems 4 and 6, if $\mathrm{r.deg}(P) \geq \mathrm{r.deg}(P_n) + 2$ or $\mathrm{r.deg}(P_n) > \mathrm{r.deg}(P) > 2$, the closed-loop system cannot be stabilized, with sufficiently small τ, no matter how C, P_n, and Q are chosen.

3. A guideline for selecting Q and P_n

The theorems in the previous section suggest some design guidelines for Q and P_n. For example, if the relative degree of the unknown plant is ensured to be less than or equal to two with known sign of high-frequency gain, then simply choose P_n such that $\mathrm{r.deg}(P_n) \geq 3$ with the same sign of high-frequency gain, and design Q with $k = 0$ and $l = \mathrm{r.deg}(P_n)$. Then, it is easily seen that all the conditions of Theorem 5 are satisfied.

The condition (iii) of Theorem 3 also allows the following interpretation. Let K_p denote the high-frequency gain of the plant $P(s)$, and its numerator and the denominator be written as $N(s) = K_p(s^{k_p} + b_p s^{k_p-1} + \cdots)$ and $D(s) = s^{l_p} + a_p s^{l_p-1} + \cdots$, respectively. Similarly, K_c, K_n, k_c, l_c, k_n, l_n, a_c, b_c, a_n, and b_n are all defined from $C(s)$ and $P_n(s)$. Moreover, let $\mu(P) =$ (sum of all zeros of P) $-$ (sum of all poles of P) $= -b_p + a_p$. Now, suppose that $\mathrm{r.deg}(P_nC) \geq 2$ and $K_nK_p > 0$ (same sign of high-frequency gains). Then, since $m_\alpha = k_p + l_n + l_c$ and $m_\beta = k_n + l_p + l_c$, it follows that $p_\alpha(s) = K_p[s^{m_\alpha} + (a_n + a_c + b_p)s^{m_\alpha-1} + \cdots]$ and $p_\beta(s) = K_n[s^{m_\beta} + (a_p + a_c + b_n)s^{m_\beta-1} + \cdots]$. Thus, (iii) becomes

$$\mu(P_n) + \frac{K_p}{K_n}\frac{a_0}{a_1 - b_1}\left(\frac{a_2 - b_2}{a_1 - b_1} - \frac{b_1}{a_0}\right) < \mu(P). \tag{5}$$

Therefore, as poles (zeros, respectively) of P_n are placed further right (left, respectively), it becomes more beneficial for robust stability. However, this may make the design of $C(s)$ more difficult since the control of stable plant is easier than that of unstable plant. It should be noted that the controller $C(s)$ does not affect (5).

3.1. A robust controller for uncertain plant with relative degree up to four

For given uncertain minimum phase plant $P(s)$, if $\mathrm{r.deg}(P) \leq 4$ and the sign of $\kappa(P)$ is known, then a robust controller can be designed, which is 'universal' in the sense that it applies to the plant of any order and of any bounded (but arbitrarily large) uncertainty. Just by reducing the parameter τ, robust stabilization is achieved.

Let $\bar{\mu}(\mathcal{P}) := \min_{P\in\mathcal{P}} \mu(P)$ and $\bar{K}_p := \max_{P\in\mathcal{P}} |\kappa(P)|$. Pick K_n such that it can be an arbitrary constant but has the same sign as $\kappa(P)$. Let $Q(s) = a_0/((\tau s)^3 + a_2(\tau s)^2 + a_1(\tau s) + a_0)$, where a_1 and a_2 are selected such that $s^2 + a_2 s + a_1 = \pi(s)$ is Hurwitz and $a_0 > 0$ is chosen sufficiently small such that $p_f(s) = s^3 + a_2s^2 + a_1s + (K_p/K_n)a_0$ is Hurwitz for all $|K_p| \leq \bar{K}_p$. In fact, it holds if $0 < a_0 < a_1a_2K_n/\bar{K}_p$, which is found, e.g., by the Routh–Hurwitz test. Now, choose the high-frequency gain of the nominal plant P_n as K_n, and select poles and zeros of P_n such that $\mathrm{r.deg}(P_n) = 3$ and

$$\mu(P_n) + \frac{\bar{K}_p}{K_n}\frac{a_0a_2}{a_1^2} < \bar{\mu}(\mathcal{P})$$

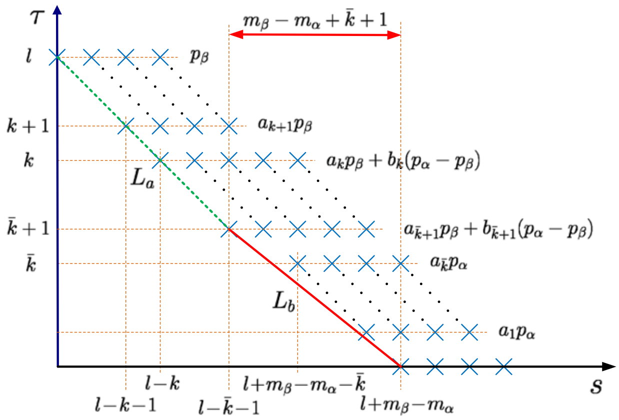

Fig. 2. Newton diagram for $\bar{\delta}(s;\tau)$ in (4) when r.deg(P) > r.deg(P_n) (i.e., $m_\beta > m_\alpha$).

is satisfied. With this P_n, C is designed such that it stabilizes P_n. The remaining freedom of choice for P_n and C can be used to satisfy given performance specifications. With the design, robust stability follows from the main theorems. (For instance, if r.deg(P) is 1 or 2, all the conditions of Theorem 5 are satisfied.)

4. Technical proofs

The conditions regarding (H1) and (H2) in all theorems follow from the same arguments as in Remark 2, which are related to the polynomial $p_\alpha(s)$. Therefore, the proof is mainly to investigate the behavior of $l + m_\beta - m_\alpha$ vanishing roots of $\bar{\delta}(s;\tau)$ in (4) and to see if they remain in the LHP while converging to the origin. The study could have been facilitated if there is no higher-order terms of τ in $\bar{\delta}(s;\tau)$ except the first order one because the classical root-locus method could be employed. However, since this is not the case, we invoke the method of Newton diagram, inspired by the higher-order root-locus method in Hahn (1981).

Proof of Theorem 4. The vanishing roots of $\bar{\delta}(s;\tau)$ have the form of $s^*(\tau) = \gamma\tau^c + o(\tau^c)$ where $o(\tau^c)$ represents the terms having higher order of τ than $c > 0$, and γ is a non-zero constant. To find c and γ, the Newton diagram[4] of $\bar{\delta}(s;\tau)$ is drawn as in Fig. 2, where it is seen that there are two groups of roots. The first group consists of $l-\bar{k}-1$ roots of the form $\gamma_a\tau^1 + o(\tau^1)$ and the second group has $m_\beta - m_\alpha + \bar{k} + 1$ roots of the form $\gamma_b\tau^{(\bar{k}+1)/(m_\beta-m_\alpha+\bar{k}+1)} + o(\cdots)$. It is also seen that γ_a and γ_b satisfy the following two equations, respectively:

$$\phi_a(\gamma) = \beta_{m_\beta}\left[\sum_{i=k+1}^{l} a_i\gamma^{l-i} + \sum_{i=\bar{k}+1}^{k}(a_i - b_i)\gamma^{l-i}\right] = 0,$$

$$\phi_b(\gamma) = (a_{\bar{k}+1} - b_{\bar{k}+1})\beta_{m_\beta} + \alpha_{m_\alpha}a_0\gamma^{m_\beta-m_\alpha+\bar{k}+1} = 0.$$

For stability, all the roots of ϕ_a and ϕ_b need to be located in LHP because they determine the location of $s^*(\tau)$ for sufficiently small τ. It is clear that a necessary condition for stability is $m_\beta - m_\alpha + \bar{k}$ is at most one, because, if not, at least one root of $\phi_b(\gamma)$ is in RHP. This explains the conditions (a) and (g). Now assuming $m_\beta - m_\alpha = 1$ and $\bar{k} = 0$, (d) ((e), respectively) implies a solution to $\phi_a(\gamma) = 0$ ($\phi_b(\gamma) = 0$, respectively) is in RHP (since $\phi_a(\gamma) = \beta_{m_\beta}\gamma^{l-1}\pi(1/\gamma)$). If $\beta_{m_\beta}/\alpha_{m_\alpha} > 0$, the second group has two roots $s^*(\tau) = \pm i\bar{\gamma}\tau^{1/2} + o(\tau^{1/2})$ where $\bar{\gamma} = \sqrt{(a_1 - b_1)\beta_{m_\beta}/(a_0\alpha_{m_\alpha})}$. With this, stability is inconclusive and we need to inspect higher order terms.

We let[5] $s^*(\tau) = (i\bar{\gamma} + \hat{s}(\tau))\tau^{1/2}$ where $\hat{s}$ is a continuous function to be found such that $\hat{s}(0) = 0$. Define $\hat{\tau} = \tau^{1/2}$ and $A(\hat{\tau}) = i\bar{\gamma} + \hat{s}(\hat{\tau}^2)$ for convenience, and regard $\bar{\delta}(s^*(\hat{\tau}^2);\hat{\tau}^2)$ as a polynomial of $\hat{s}$ with the parameter $\hat{\tau}$, that is, from (4),

$$\begin{aligned}\bar{\delta}(s^*(\hat{\tau}^2);\hat{\tau}^2) &= a_0\big(\alpha_{m_\alpha}A^{l+1}\hat{\tau}^{l+1} + \alpha_{m_\alpha-1}A^{l+2}\hat{\tau}^{l+2} + \cdots\big)\\ &\quad + (a_1 - b_1)\big(\beta_{m_\beta}A^{l-1}\hat{\tau}^{l+1} + \beta_{m_\beta-1}A^{l}\hat{\tau}^{l+2} + \cdots\big)\\ &\quad + b_1\big(\alpha_{m_\alpha}A^{l}\hat{\tau}^{l+2} + \alpha_{m_\alpha-1}A^{l+1}\hat{\tau}^{l+3} + \cdots\big)\\ &\quad + (a_2 - b_2)\big(\beta_{m_\beta}A^{l-2}\hat{\tau}^{l+2} + \beta_{m_\beta-1}A^{l-1}\hat{\tau}^{l+3} + \cdots\big)\\ &\quad + b_2\big(\alpha_{m_\alpha}A^{l-1}\hat{\tau}^{l+3} + \cdots\big) + \cdots =: \hat{\delta}(\hat{s};\hat{\tau}).\end{aligned}$$

Collecting the terms in increasing order of $\hat{\tau}$, it becomes

$$\begin{aligned}\hat{\delta}(\hat{s};\hat{\tau}) &= \hat{\tau}^{l+1}\big[a_0\alpha_{m_\alpha}A^{l+1} + (a_1 - b_1)\beta_{m_\beta}A^{l-1}\big]\\ &\quad + \hat{\tau}^{l+2}\big[a_0\alpha_{m_\alpha-1}A^{l+2} + ((a_1 - b_1)\beta_{m_\beta-1} + b_1\alpha_{m_\alpha})A^{l}\\ &\quad + (a_2 - b_2)\beta_{m_\beta}A^{l-2}\big] + \hat{\tau}^{l+3}[\cdots] + \cdots.\end{aligned}$$

By expanding with $A = i\bar{\gamma} + \hat{s}(\hat{\tau}^2)$, it is seen that the constant term (with respect to $\hat{s}$) in the coefficient of $\hat{\tau}^{l+1}$ (the lowest power of $\hat{\tau}$) is zero by the definition of $\bar{\gamma}$. With this fact, the Newton diagram of $\hat{\delta}(\hat{s};\hat{\tau})/\hat{\tau}^{l+1}$ (Fig. 3) suggests that it has one root $\hat{s}^*(\hat{\tau})$ of the form $\hat{\gamma}\hat{\tau}^1 + o(\hat{\tau}^1)$ and $\hat{\gamma}$ is the root of

$$\begin{aligned}\hat{\phi}(\hat{\gamma}) = &\left(\frac{\alpha_{m_\alpha-1}\beta_{m_\beta}^2(a_1 - b_1)^2}{\alpha_{m_\alpha}^2 a_0} - \frac{(a_1 - b_1)^2\beta_{m_\beta}\beta_{m_\beta-1}}{\alpha_{m_\alpha}a_0}\right.\\ &\left. - \frac{(a_1 - b_1)b_1\beta_{m_\beta}}{a_0} + (a_2 - b_2)\beta_{m_\beta}\right)\\ &- 2(a_1 - b_1)\beta_{m_\beta}\hat{\gamma}.\end{aligned}$$

The condition (f) implies that $\hat{\gamma}$ is in RHP, and so is $s^*(\tau) = i\bar{\gamma}\tau^{1/2} + \hat{\gamma}\tau^1 + o(\tau^1)$ as $\tau \to 0$.

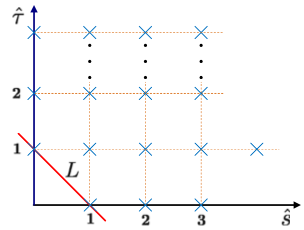

Fig. 3. Newton diagram for $\hat{\delta}(\hat{s};\hat{\tau})/\hat{\tau}^{l+1}$.

Proof of Theorem 3. Conclusions of Theorem 3 are easily derived from the proof of Theorem 4. Indeed, by the condition (i), it follows that $a_1 - b_1 > 0$ and $\bar{k} = 0$, which yields $\phi_b(\gamma) = (a_1 - b_1)\beta_{m_\beta}$

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[4] The non-zero coefficient of the term $\tau^j s^i$ is marked as $\times$ in the coordinate (i, j). Then, a convex hull of all marked $\times$ is considered, and the line segments with different slopes, located on the boundary in the lower-left side, are found. (L_a and L_b in Fig. 2.) Let N be the number of such line segments. From the figure, the following facts are read out: (i) the total number of roots converging to zero as $\tau \to 0$ is the index of the leftmost $\times$ in the row of τ^0 (which is $l + m_\beta - m_\alpha$ in Fig. 2). (ii) These roots are divided by N groups. (iii) For each group, there is m roots of the form $s_i^*(\tau) = \gamma_i\tau^c + o(\tau^c)$, $1 \le i \le m$, where $c = -$(slope of the line segment) and m is the difference between the horizontal indices of the rightmost mark and the leftmost mark in the line segment. (iv) The value of γ_i is determined by finding roots of the m-th order polynomial $\phi(\gamma)$ whose coefficients are the values of those marks that touch the corresponding line segment.

[5] As for the case where $s^*(\tau) = (-i\bar{\gamma} + \hat{s}(\tau))\tau^{1/2}$, the same conclusion is obtained and the details are omitted.

$+ \alpha_{m_\alpha} a_0 \gamma^2$. Then, the conditions (ii) and (iii) ((i), respectively) imply that all the roots of ϕ_b (ϕ_a, respectively) are located in LHP.

Proof of Theorems 5 and 6. The proofs have to be omitted due to page limitation. They are however available upon request, or at http://hdl.handle.net/10371/91261.

5. Concluding remarks

We have studied the robust stability of the DOB control system when the relative degree of the plant is not the same as that of the nominal model. By introducing the method of the Newton diagram, whose utility became apparent in conjunction with stability analysis of DOB, the following facts are revealed under the assumption of small time constant of Q-filter. If $\mathrm{r.deg}(P) = \mathrm{r.deg}(P_\mathrm{n}) + 1$, robust stability can be achieved by an appropriate design of P_n as well as Q. If $1 \leq \mathrm{r.deg}(P) \leq 2$, then robust stability is always achievable. If $\mathrm{r.deg}(P) \geq \mathrm{r.deg}(P_\mathrm{n}) + 2$ or $\mathrm{r.deg}(P_\mathrm{n}) > \mathrm{r.deg}(P) > 2$, then robust stabilization is not possible no matter how P_n, C, and Q are selected. The lesson of the study is that one needs to estimate the relative degree of the plant as close as possible, because, if not, robust stability may not be achievable with sufficiently small time constant of the Q-filter.

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