Applied Mathematics and Computation 485 (2025) 129022

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Full Length Article

Laplacian eigenvalue distribution for unicyclic graphs

Sunyo Moon [a], Seungkook Park [b,*]

[a] School of Computational Sciences, Korea Institute for Advanced Study, 85 Hoegiro Dongdaemun-gu, Seoul, 02455, Republic of Korea
[b] Department of Mathematics and Research Institute of Natural Sciences, Sookmyung Women's University, Cheongpa-ro 47-gil 100, Yongsan-gu, Seoul, 04310, Republic of Korea

ARTICLE INFO

MSC:
15A18
05C50

Keywords:
Laplacian eigenvalues
Unicyclic graph
Diameter
Girth

ABSTRACT

Let G be a graph and let $m_G[0,1)$ denote the number of Laplacian eigenvalues of G in the interval $[0,1)$. For a tree T with diameter d, Guo, Xue, and Liu proved that $m_T[0,1) \geq (d+1)/3$. In this paper, we provide a lower bound for $m_G[0,1)$ when G is a unicyclic graph, in terms of the diameter and girth of G. Moreover, for the lollipop graph, under certain conditions on its diameter and girth, we give a formula for the exact value of $m_G[0,1)$.

1. Introduction

Let $G=(V(G),E(G))$ be a simple graph, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. We will denote the adjacency matrix of G by $A(G)$. The *degree* of the vertex v is the number of vertices adjacent to v. The Laplacian matrix $L(G)$ of G is defined by $L(G)=D(G)-A(G)$, where $D(G)$ is the diagonal matrix of vertex degree. An eigenvalue of $L(G)$ is called a *Laplacian eigenvalue* of G. Since $L(G)$ is symmetric and positive semidefinite, all its eigenvalues are real and non-negative. We shall index the Laplacian eigenvalues of G in non-decreasing order and denote them as

$$0=\mu_1(G)\leq \mu_2(G)\leq \cdots \leq \mu_n(G).$$

The multiplicity of the Laplacian eigenvalue μ is denoted by $m_G(\mu)$. It is well known that the largest Laplacian eigenvalue can not exceed n. Thus, we only consider the intervals within $[0,n]$. We denote the number of eigenvalues of $L(G)$ within the interval I by $m_G I$. In a connected graph G, the distance between two vertices is defined as the shortest path length that includes them. The *diameter* of G is the largest distance between any two vertices in G, which is denoted by $d(G)$. A path whose length is equal to the diameter of the graph is called a *diametral path*. A *dominating set* is a subset D of $V(G)$ such that every vertex $u \in V(G)\backslash D$ is adjacent to a vertex in D. The *domination number* of G is the minimum cardinality of a dominating set of G, which is denoted by $\gamma(G)$. A *tree* is a connected graph without any cycles and a *unicyclic graph* is a connected graph containing exactly one cycle. For a graph G, a lower bound for the domination number $\gamma(G)$ was given in [1,2], which is

$$\frac{d(G)+1}{3}\leq \gamma(G).$$

* Corresponding author.
E-mail addresses: symoon@kias.re.kr (S. Moon), skpark@sookmyung.ac.kr (S. Park).

https://doi.org/10.1016/j.amc.2024.129022
Received 19 October 2023; Received in revised form 12 August 2024; Accepted 18 August 2024
Available online 26 August 2024

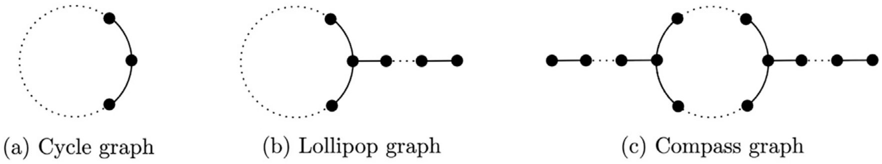

Fig. 1. Three types of unicyclic graphs.

In 2016, Hedetniemi, Jacobs, and Trevisan [3] proved that

$$m_G[0,1) \leq \gamma(G)$$

for any graph G. Recently, Guo, Xue, and Liu [4] established a relation between $\frac{d(G)+1}{3}$ and $m_G[0,1)$ for the case where G is a tree. They showed that for any tree T, the inequality

$$\frac{d(T)+1}{3} \leq m_T[0,1)$$

is satisfied. Hence we have

$$\frac{d(T)+1}{3} \leq m_T[0,1) \leq \gamma(T).$$

However, the inequality $\frac{d(G)+1}{3} \leq m_G[0,1)$ does not hold for a general connected graph G. For example, the cycle graph C_6 with 6 vertices has one Laplacian eigenvalue in $[0,1)$. However, we have $\frac{d(C_6)+1}{3} = \frac{4}{3} > 1$. In this paper, we show that if G is unicyclic with girth g, then

$$\left\lceil \frac{d(G)}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 \leq m_G[0,1).$$

Hence, for a unicyclic graph G with girth $g \geq 7$, we obtain the following relation:

$$\frac{d(G)+1}{3} \leq \left\lceil \frac{d(G)}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 \leq m_G[0,1) \leq \gamma(G).$$

2. Preliminaries

Let G be a graph and let e be an edge of G. The graph $G - e$ is obtained from G by deleting the edge e. The following theorem implies that removing an edge from a graph leads to a decrease in the values of the Laplacian eigenvalues of the graph.

Theorem 2.1 *([5]). Let G be a graph with n vertices and let e be an edge of G. For $i = 1, \ldots, n-1$,*

$$\mu_i(G) \leq \mu_{i+1}(G-e) \leq \mu_{i+1}(G).$$

The following lemma shows that attaching a pendant vertex to a graph does not change the lower bound for the number of Laplacian eigenvalues in $[0,1)$.

Lemma 2.2 *([4]). Let G be a graph with n vertices and let G' be a graph obtained from G by deleting a pendant vertex. If $m_{G'}[0,1) \geq k$ then $m_G[0,1) \geq k$.*

Let G be unicyclic with n vertices and girth $g \geq 3$. Then G has n edges and consists of a cycle C_g of length g $(3 \leq g \leq n)$ and disjoint maximal trees $T_1, \ldots, T_m$ $(0 \leq m \leq g)$ such that each tree has exactly one vertex in common with C_g. Thus, any unicyclic graph G can be obtained by attaching pendant vertices to the minimal unicyclic subgraph of G that contains a diametral path of G. By Lemma 2.2, it is enough to consider the minimal unicyclic subgraph of G that contains a diametral path of G, which are given in Fig. 1. The *lollipop graph* is obtained by appending a vertex of the cycle C_g to an end vertex of the path P_{n-g}. The lollipop graph with n vertices and girth g is denoted by $C_{n,g}$. In order to define the compass graph, we label the n vertices of the graph as shown in Fig. 2. Let t and s be positive integers such that $t + s = n - g$. A compass graph consists of a cyclic graph C_g with paths P_t and P_s attached. To be more precise, the *compass graph* is obtained by attaching an end vertex of the path P_t to a cycle vertex of the lollipop graph $C_{n-t,g}$ such that its distance from the degree 3 vertex is g'. If $g' = 0$, then the end vertex of P_t is attached to the degree 3 cycle vertex of $C_{n-t,g}$ (see Fig. 2 (b)). We denote this graph by $C_{n,g}(g', t)$. Note that the diameter of $C_{n,g}(g', t)$ is $d(C_{n,g}(g', t)) = g' + t + s$.

Let G be a unicyclic graph and let G' be the minimal unicyclic subgraph of G that contains a diametral path of G. We will illustrate how to deduce G' from G and conclude that it is sufficient to find the lower bound of $m_{G'}[0,1)$.

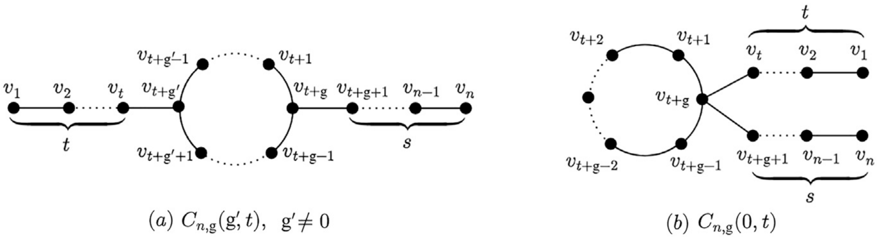

Fig. 2. The compass graph $C_{n,g}(g', t)$.

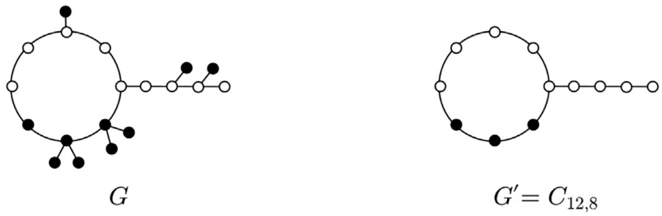

Fig. 3. Reducing to lollipop graph.

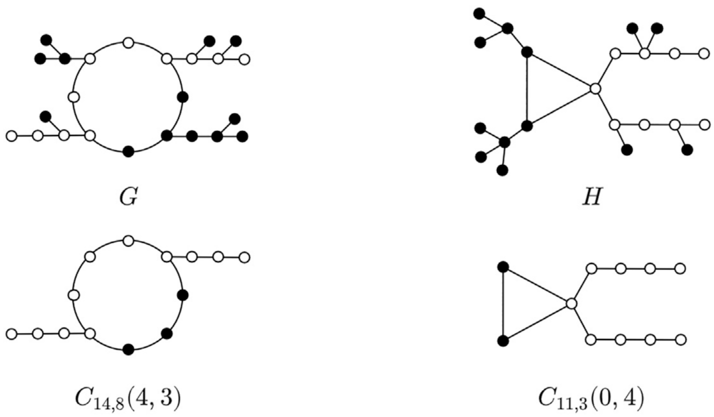

Fig. 4. Reducing to compass graphs.

Example 2.3. Let G be a graph with 19 vertices as shown on the left in Fig. 3. The diameter of G is 8 and the girth of G is 8. The path consisting of the white vertices is the diametral path of G. Then G' is the lollipop graph $C_{12,8}$. Suppose that $m_{C_{12,8}}[0,1) \geq k$. Then, by Lemma 2.2, $m_G[0,1) \geq k$. Our bound for $m_{C_{12,8}}[0,1)$ is

$$m_{C_{12,8}}[0,1) \geq \left\lceil \frac{8}{3} \right\rceil + \left\lceil \frac{8}{6} \right\rceil - 1 = 4.$$

Hence $m_G[0,1) \geq 4$. In fact, $m_{C_{12,8}}[0,1) = 4$ and $m_G[0,1) = 6$.

Example 2.4. Let G and H be unicyclic graphs in Fig. 4. The minimal unicyclic subgraphs containing a diametral path of G and H are the compass graphs $C_{14,8}(4,3)$ and $C_{11,3}(0,4)$ in Fig. 4, respectively. The diametral paths are represented by the white vertices. The bounds for $m_{C_{14,8}(4,3)}[0,1)$ and $m_{C_{11,3}(0,4)}[0,1)$ are

$$m_{C_{14,8}(4,3)}[0,1) \geq \left\lceil \frac{10}{3} \right\rceil + \left\lceil \frac{8}{6} \right\rceil - 1 = 5$$

and

$$m_{C_{11,3}(0,4)}[0,1) \geq \left\lceil \frac{8}{3} \right\rceil + \left\lceil \frac{3}{6} \right\rceil - 1 = 3.$$

Hence, we conclude that $m_G[0,1) \geq 5$ and $m_H[0,1) \geq 3$. We note that $m_G[0,1) = 8$ and $m_H[0,1) = 6$. Also, the bounds for $m_{C_{14,8}(4,3)}[0,1)$ and $m_{C_{11,3}(0,4)}[0,1)$ are sharp.

Now, we list some facts and lemmas which are used to prove our main results. For a square matrix M, we denote $\mu_i(M)$ as the ith smallest eigenvalue of M.

Lemma 2.5 *([6]). Let A and B be $n \times n$ Hermitian matrices. For the integers i and j $(1 \le i, j \le n)$ satisfying $1 \le i + j - n \le n$, we have*

$$\mu_{i+j-n}(A+B) \le \mu_i(A) + \mu_j(B).$$

Moreover, equality holds if and only if there exists a unit vector x such that $Ax = \mu_i(A)x$, $Bx = \mu_j(B)x$, and $(A+B)x = \mu_{i+j-n}(A+B)x$.

Lemma 2.6 *([5]). Let G be a graph with n vertices and suppose that H is a graph obtained from G and P_3 by joining a pendant vertex of P_3 to a vertex of G. Then $m_H(1) = m_G(1)$.*

Corollary 2.7. *Let G be a graph with n vertices and let m be a positive integer divisible by 3. Suppose that H is a graph obtained from G and P_m by joining a pendant vertex of P_m to a vertex of G. Then $m_H(1) = m_G(1)$.*

Remark 2.8. Since the Laplacian eigenvalues of P_n are

$$2 - 2\cos\frac{k\pi}{n},$$

where $k = 0, \dots, n-1$, we have

(a) $m_{P_n}[0,1) = \left\lceil \frac{n}{3} \right\rceil$,
(b) $3 \mid n$ if and only if $m_{P_n}(1) = 1$.

Lemma 2.9. *Let P_n be the path graph with n vertices and let $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ be a vector of order n, where*

$$x_i = \begin{cases} 1, & \text{if } i \equiv 1,\ 6 \pmod 6, \\ 0, & \text{if } i \equiv 2,\ 5 \pmod 6, \\ -1, & \text{if } i \equiv 3,\ 4 \pmod 6. \end{cases}$$

If n is divisible by 3, then the vector x is an eigenvector corresponding to the Laplacian eigenvalue 1 of P_n.

Proof. One can immediately verify that $L(P_n)x = x$. □

We denote the characteristic polynomial of a matrix M by $\phi(M) = \phi(M; x) = \det(xI - M)$, where I is the identity matrix. If $M = L(G)$, then we will write $\phi(L(G))$ as $\phi(G)$. For a vertex v of G, let $L_v(G)$ be the principal submatrix of $L(G)$ formed by deleting the row and column corresponding to the vertex v. Let B_n be the matrix of order n obtained from $L(P_{n+1})$ by deleting the row and column corresponding to one of the end vertices of P_{n+1}. Let H_n be the matrix of order n obtained from $L(P_{n+2})$ by deleting the rows and columns corresponding to both end vertices of P_{n+2}.

Lemma 2.10 *([7]). Let G be the graph obtained by joining the vertex u of the graph G_1 to the vertex v of the graph G_2 by an edge. Then*

$$\phi(G) = \phi(G_1)\phi(G_2) - \phi(G_1)\phi(L_v(G_2)) - \phi(G_2)\phi(L_u(G_1)).$$

Lemma 2.11 *([8]). Set $\phi(P_0) = 0$, $\phi(B_0) = 1$, $\phi(H_0) = 1$. We have*

(a) $x\phi(B_n) = \phi(P_{n+1}) + \phi(P_n)$;
(b) $\phi(P_n) = x\phi(H_{n-1})$, $(n \ge 1)$;
(c) $\phi(C_n) = \frac{1}{x}\phi(P_{n+1}) - \frac{1}{x}\phi(P_{n-1}) + 2(-1)^{n+1}$, $(n \ge 3, x \ne 0)$.

Lemma 2.12. *Let $C_{n,g}$ be a lollipop graph with n vertices and girth g. Then the characteristic polynomial of $C_{n,g}$ can be represented as*

$$\phi(C_{n,g}) = \phi(P_{n-g})\left(\phi(C_g) - \frac{1}{x}\phi(C_g)\right) - \frac{1}{x}\phi(C_g)\phi(P_{n-g-1}) - \frac{1}{x}\phi(P_{n-g})\phi(P_g).$$

Proof. Let u be a vertex of C_g and let v be an end vertex of P_{n-g}. Then, by Lemma 2.10,

$$\phi(C_{n,g}) = \phi(C_g)\phi(P_{n-g}) - \phi(C_g)\phi(B_{n-g-1}) - \phi(P_{n-g})\phi(H_{g-1}).$$

By Lemma 2.11 (a) and (b), we have

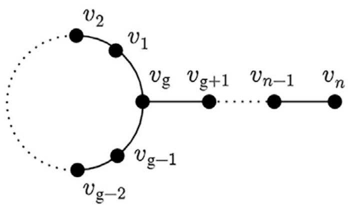

Fig. 5. Lollipop graph $C_{n,g}$.

$$\begin{aligned}\phi(C_{n,g}) &= \phi(C_g)\phi(P_{n-g}) - \frac{1}{x}\phi(C_g)\big(\phi(P_{n-g}) + \phi(P_{n-g-1})\big) - \frac{1}{x}\phi(P_{n-g})\phi(P_g) \\ &= \phi(P_{n-g})\left(\phi(C_g) - \frac{1}{x}\phi(C_g)\right) - \frac{1}{x}\phi(C_g)\phi(P_{n-g-1}) - \frac{1}{x}\phi(P_{n-g})\phi(P_g). \quad \square\end{aligned}$$

3. Main results

In this section, we establish a lower bound for $m_G[0,1)$, where G represents the cycle, lollipop, and compass graphs. For the cycle graph, the exact value of $m_G[0,1)$ can be computed, as shown in the following remark.

Remark 3.1. Since the Laplacian eigenvalues of C_n are

$$2 - 2\cos\frac{2\pi k}{n},$$

where $k = 0, \ldots, n-1$, we have

(a) $m_{C_n}[0,1) = 2\left\lceil \frac{n}{6} \right\rceil - 1$,
(b) $6 \mid n$ if and only if $m_{C_n}(1) = 2$.

By Remark 3.1, we have $m_{C_n}[0,1) = 2\lceil \frac{n}{6} \rceil - 1$. Since the diameter of C_n is $\lfloor \frac{n}{2} \rfloor$, we obtain

$$2\left\lceil \frac{n}{6} \right\rceil - 1 - \left(\left\lceil \frac{1}{3}\left\lfloor \frac{n}{2} \right\rfloor \right\rceil + \left\lceil \frac{n}{6} \right\rceil - 1\right) = \left\lceil \frac{n}{6} \right\rceil - \left\lceil \frac{1}{3}\left\lfloor \frac{n}{2} \right\rfloor \right\rceil \geq \left\lceil \frac{n}{6} \right\rceil - \left\lceil \frac{1}{3}\left\lceil \frac{n}{2} \right\rceil \right\rceil = 0.$$

Thus

$$m_{C_n}[0,1) \geq \left\lceil \frac{d(C_n)}{3} \right\rceil + \left\lceil \frac{n}{6} \right\rceil - 1.$$

3.1. Lollipop graphs

We consider the lollipop graph $C_{n,g}$ with n vertices and girth g. We label the vertices as shown in Fig. 5. Then the diameter of $C_{n,g}$ is $d(C_{n,g}) = \lfloor \frac{g}{2} \rfloor + (n-g) = n - \lceil \frac{g}{2} \rceil$. Note that $m_{G_1 \cup G_2}(\mu) = m_{G_1}(\mu) + m_{G_2}(\mu)$ for any disjoint graphs G_1 and G_2, where μ is a Laplacian eigenvalue of $G_1 \cup G_2$.

Proposition 3.2. *Let $C_{n,g}$ be the lollipop graph with n vertices and girth g. Then*

$$m_{C_{n,g}}[0,1) \geq \left\lceil \frac{d(C_{n,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1.$$

Proof. Let $k = \lceil \frac{d(C_{n,g})}{3} \rceil + \left\lceil \frac{g}{6} \right\rceil - 1$. We consider the following two subgraphs:

$$P_n = C_{n,g} - v_1 v_g \quad \text{and} \quad C_g \cup P_{n-g} = C_{n,g} - v_g v_{g+1},$$

where v_1, v_g, v_{g+1} are labeled as in Fig. 5.

Case 1. Suppose that $g \equiv 1 \pmod 6$. Then $\lceil \frac{g-1}{2} \rceil = \lfloor \frac{g}{2} \rfloor$. We have

$$\begin{aligned}m_{C_g \cup P_{n-g}}[0,1) &= m_{C_g}[0,1) + m_{P_{n-g}}[0,1) \\ &= 2\left\lceil \frac{g}{6} \right\rceil - 1 + \left\lceil \frac{n-g}{3} \right\rceil \\ &= \left\lceil \frac{n-g}{3} \right\rceil + \left\lceil \frac{g-1}{6} \right\rceil + \left\lceil \frac{g}{6} \right\rceil\end{aligned}$$

$$\geq \left\lceil \frac{n-g+\left\lceil \frac{g-1}{2} \right\rceil}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil$$

$$= \left\lceil \frac{n-g+\lfloor \frac{g}{2} \rfloor}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil$$

$$= \left\lceil \frac{d(C_{n,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil = k+1.$$

By Lemma 2.1, we have $\mu_k(C_{n,g}) \leq \mu_{k+1}(C_g \cup P_{n-g}) < 1$. Thus $m_{C_{n,g}}[0,1) \geq k$.

Case 2. Suppose that $g \equiv 2 \pmod 6$. Then $\lceil \frac{g}{2} \rceil = \lfloor \frac{g}{2} \rfloor$. Note that

$$\begin{aligned} m_{C_g \cup P_{n-g}}[0,1) &= 2\left\lceil \frac{g}{6} \right\rceil - 1 + \left\lceil \frac{n-g}{3} \right\rceil \\ &= \left\lceil \frac{n-g}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 \\ &\geq \left\lceil \frac{n-g+\lceil \frac{g}{2} \rceil}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 \\ &= \left\lceil \frac{n-g+\lfloor \frac{g}{2} \rfloor}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 \\ &= \left\lceil \frac{d(C_{n,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 = k. \end{aligned}$$

If $n-g \not\equiv 0 \pmod 3$, then $m_{C_g \cup P_{n-g}}[0,1) > k$ since $\left\lceil \frac{n-g}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil > \left\lceil \frac{n-g+\lfloor \frac{g}{2} \rfloor}{3} \right\rceil$. Thus $\mu_k(C_{n,g}) \leq \mu_{k+1}(C_g \cup P_{n-g}) < 1$ and hence $m_{C_{n,g}}[0,1) \geq k$. Now, we consider the case $n-g \equiv 0 \pmod 3$. Since $\left\lceil \frac{n-g}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil = \left\lceil \frac{n-g+\lfloor \frac{g}{2} \rfloor}{3} \right\rceil$, it follows that $m_{C_g \cup P_{n-g}}[0,1) = k$. Suppose that $m_{C_{n,g}}[0,1) < k$. Then $\mu_k(C_{n,g}) \geq 1$. Since $n-g \equiv 0 \pmod 3$, P_{n-g} has a Laplacian eigenvalue 1. Hence $\mu_{k+1}(C_g \cup P_{n-g}) = 1$. It follows that $\mu_k(C_{n,g}) = \mu_{k+1}(C_g \cup P_{n-g}) = 1$ by Lemma 2.1. Note that $L(C_{n,g}) - L(C_g \cup P_{n-g}) = M = (m_{ij})$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \{(g,g),\ (g+1,g+1)\}, \\ -1, & \text{if } (i,j) \in \{(g,g+1),\ (g+1,g)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since M is permutationally similar to $L(K_2 \cup (n-2)K_1)$, we have

$$\mu_1(M) = \cdots = \mu_{n-1}(M) = 0 \text{ and } \mu_n(M) = 2.$$

Then $\mu_k(C_{n,g}) = \mu_{k+1}(C_g \cup P_{n-g}) + \mu_{n-1}(M)$. By Lemma 2.5, there exists an eigenvector z such that

$$\begin{aligned} &L(C_{n,g})z = \mu_k(C_{n,g})z = z, \\ &L(C_g \cup P_{n-g})z = \mu_{k+1}(C_g \cup P_{n-g})z = z, \text{ and} \\ &Mz = \mu_{n-1}(M)z = \mathbf{0}. \end{aligned}$$

Let x be the eigenvector corresponding to the Laplacian eigenvalue 1 of P_{n-g}, which is given in Lemma 2.9. Since 1 is not a Laplacian eigenvalue of C_g, the eigenvector z corresponding to the Laplacian eigenvalue 1 of $C_g \cup P_{n-g}$ is a multiple of

$$\left[\frac{\mathbf{0}}{x} \right].$$

Since the gth coordinate of z is 0 and the $(g+1)$th coordinate of z is nonzero, we have $Mz \neq 0$. This is a contradiction and hence $m_{C_{n,g}}[0,1) \geq k$.

Case 3. Suppose that $g \equiv 3,4 \pmod 6$. Then $\lceil \frac{g}{2} \rceil \equiv 2 \pmod 3$. If $n \not\equiv 0 \pmod 3$, then

$$\begin{aligned} m_{P_n}[0,1) &= \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{n+1}{3} \right\rceil \\ &= \left\lceil \frac{d(C_{n,g}) + \lceil \frac{g}{2} \rceil + 1}{3} \right\rceil \end{aligned}$$

$$= \left\lceil \frac{d(C_{n,g})}{3} \right\rceil + \left\lceil \frac{\lceil \frac{g}{2} \rceil + 1}{3} \right\rceil$$
$$= \left\lceil \frac{d(C_{n,g})}{3} \right\rceil + \left\lceil \frac{g+2}{6} \right\rceil$$
$$= \left\lceil \frac{d(C_{n,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil = k+1.$$

Thus $\mu_k(C_{n,g}) \le \mu_{k+1}(P_n) < 1$ and hence $m_{C_{n,g}}[0,1) \ge k$.

Now, we assume that $n = d(C_{n,g}) + \lceil \frac{g}{2} \rceil \equiv 0 \pmod 3$. Then $d(C_{n,g}) \equiv 1 \pmod 3$. Suppose that $m_{C_{n,g}}[0,1) < k$. Then $\mu_k(C_{n,g}) \ge 1$. Since

$$m_{P_n}[0,1) = \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{d(C_{n,g}) + \lceil \frac{g}{2} \rceil}{3} \right\rceil = \left\lceil \frac{d(C_{n,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 = k,$$

we have $\mu_{k+1}(P_n) = 1$ and hence $\mu_k(C_{n,g}) = 1$. Note that $L(C_{n,g}) - L(P_n) = N = (n_{ij})$, where

$$n_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \{(1,1),\ (g,g)\}, \\ -1, & \text{if } (i,j) \in \{(1,g),\ (g,1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since N is permutationally similar to $L(K_2 \cup (n-2)K_1)$, we have

$$\mu_1(N) = \cdots = \mu_{n-1}(N) = 0 \ \text{ and } \ \mu_n(N) = 2.$$

Then $\mu_k(C_{n,g}) = \mu_{k+1}(P_n) + \mu_{n-1}(N)$. By Lemma 2.5, there exists an eigenvector x such that

$$L(C_{n,g})x = \mu_k(C_{n,g})x = x,$$
$$L(P_n)x = \mu_{k+1}(P_n)x = x, \ \text{ and}$$
$$Nx = \mu_{n-1}(N)x = \mathbf{0}.$$

Let x be the eigenvector corresponding to the Laplacian eigenvalue 1 of P_n, which is given in Lemma 2.9. Then the first coordinate of x is 1. Since $g \equiv 3,4 \pmod 6$, the gth coordinate of x is -1. Hence $Nx \ne \mathbf{0}$. This is a contradiction and hence $m_{C_{n,g}}[0,1) \ge k$.

Case 4. Suppose that $g \equiv 0,5 \pmod 6$. Then $\lceil \frac{g}{2} \rceil \equiv 0 \pmod 3$. Since

$$m_{P_n}[0,1) = \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{d(C_{n,g}) + \lceil \frac{g}{2} \rceil}{3} \right\rceil = \left\lceil \frac{d(C_{n,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil = k+1,$$

we obtain $\mu_k(C_{n,g}) \le \mu_{k+1}(P_n) < 1$. Thus $m_{C_{n,g}}[0,1) \ge k$. □

The following proposition shows that for the special case $d(C_{n,g}) \equiv 0 \pmod 3$ and $g \not\equiv 0 \pmod 6$, we can find the exact number of Laplacian eigenvalues in the interval $[0,1)$.

Proposition 3.3. *Let $C_{n,g}$ be the lollipop graph with n vertices and girth g. If $d(C_{n,g}) \equiv 0 \pmod 3$ and $g \not\equiv 0 \pmod 6$ then*

$$m_{C_{n,g}}[0,1) = \frac{d(C_{n,g})}{3} + \left\lceil \frac{g}{6} \right\rceil.$$

Proof. For simplicity of notation, we write d instead of $d(C_{n,g})$. Let $k = \frac{d}{3} + \lceil \frac{g}{6} \rceil$. Then

$$k = \frac{d}{3} + \left\lceil \frac{g}{6} \right\rceil = \left\lceil \frac{d + \lceil \frac{g}{2} \rceil}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil.$$

By deleting edges $v_1 v_g$ and $v_{\lceil \frac{g}{2} \rceil} v_{\lceil \frac{g}{2} \rceil + 1}$ from G, we obtain a subgraph $P_d \cup P_{\lceil \frac{g}{2} \rceil}$ (see Fig. 6). Note that $m_{P_d \cup P_{\lceil \frac{g}{2} \rceil}}[0,1) = k$. Since $d \equiv 0 \pmod 3$, we obtain $\mu_{k+1}(P_d \cup P_{\lceil \frac{g}{2} \rceil}) = 1$. By Lemma 2.1, we have

$$\mu_{k-1}(C_{n,g}) \le \mu_k(P_n) \le \mu_k(C_{n,g}) \le \mu_{k+1}(P_n) \le \mu_{k+1}(C_{n,g})$$

and

$$\mu_k(P_n) \le \mu_{k+1}(P_d \cup P_{\lceil \frac{g}{2} \rceil}) \le \mu_{k+1}(P_n) \le \mu_{k+2}(P_d \cup P_{\lceil \frac{g}{2} \rceil}).$$

In order to prove $m_{C_{n,g}}[0,1) = k$, it suffices to show that

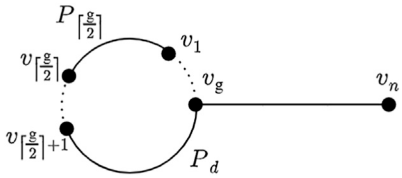

Fig. 6. The subgraph $P_d \cup P_{\lceil \frac{g}{2} \rceil}$ of $C_{n,g}$.

$$\mu_k(C_{n,g}) < \mu_{k+1}(P_d \cup P_{\lceil \frac{g}{2} \rceil}) = 1.$$

If $\mu_k(P_n) = \mu_k(C_{n,g})$, then $\mu_k(C_{n,g}) = \mu_k(P_n) < 1$ since $m_{P_n}[0,1) = k$. Suppose that $\mu_k(P_n) \neq \mu_k(C_{n,g})$. Let α be a real number such that

$$\mu_{k-1}(C_{n,g}) < \alpha < \mu_k(C_{n,g}).$$

If the signs of the values $\phi(C_{n,g};\alpha)$ and $\phi(C_{n,g};1)$ are opposite, then

$$\alpha < \mu_k(C_{n,g}) < \mu_{k+1}(P_d \cup P_{\lceil \frac{g}{2} \rceil}) = 1.$$

Thus we only need to show that the signs of the values $\phi(C_{n,g};\alpha)$ and $\phi(C_{n,g};1)$ are opposite. First, we determine the sign of the value $\phi(C_{n,g};\alpha)$. Suppose that $g \equiv 1,2 \pmod 6$. Then $n = d + \lceil \frac{g}{2} \rceil \equiv 1 \pmod 3$. Let $n = 3t+1$ for some positive integer t. Then $k = \lceil \frac{n}{3} \rceil = t+1$. Hence n and k are either both even or both odd. By considering the shape of the graph of $y = \phi(C_{n,g};x)$, we deduce that the value of $\phi(C_{n,g};\alpha)$ is negative for $\mu_{k-1}(C_{n,g}) < \alpha < \mu_k(C_{n,g})$. Similarly, if $g \equiv 5 \pmod 6$, then the value of $\phi(C_{n,g};\alpha)$ is negative. Suppose that $g \equiv 3,4 \pmod 6$. Then $n = d + \lceil \frac{g}{2} \rceil \equiv 2 \pmod 3$. Let $n = 3t+2$ for some positive integer t. Then $k = \lceil \frac{n}{3} \rceil = t+1$. Thus, n and k have different parity. By considering the shape of the graph of $y = \phi(C_{n,g};x)$, we deduce that the value of $\phi(C_{n,g};\alpha)$ is positive for $\mu_{k-1}(C_{n,g}) < \alpha < \mu_k(C_{n,g})$.

Now, we determine the sign of the value $\phi(C_g;1)$. By Lemma 2.11 (c) and Lemma 2.12,

$$\phi(C_{n,g};1) = -\big(\phi(P_{g+1};1) - \phi(P_{g-1};1) + 2(-1)^{g+1}\big)\phi(P_{n-g-1};1) - \phi(P_{n-g};1)\phi(P_g;1).$$

Since $\phi(P_0) = 0$, $\phi(P_1;1) = 1$ and $\phi(P_{n+1}) = (x-2)\phi(P_n) - \phi(P_{n-1})$, we have

$$\phi(P_n;1) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3, \\ 1, & \text{if } n \equiv 1 \pmod 3, \\ -1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Since $n - g = d - \lfloor \frac{g}{2} \rfloor \equiv -\lfloor \frac{g}{2} \rfloor \pmod 3$, we have

$$\phi(C_{n,g};1) = -\big(\phi(P_{g+1};1) - \phi(P_{g-1};1) + 2(-1)^{g+1}\big)\phi(P_{-\lfloor \frac{g}{2} \rfloor-1};1) - \phi(P_{-\lfloor \frac{g}{2} \rfloor};1)\phi(P_g;1).$$

Then

$$\phi(C_{n,g};1) = \begin{cases} 1, & \text{if } g \equiv 1 \pmod 6, \\ 2, & \text{if } g \equiv 2 \pmod 6, \\ -4, & \text{if } g \equiv 3 \pmod 6, \\ -1, & \text{if } g \equiv 4 \pmod 6, \\ 1, & \text{if } g \equiv 5 \pmod 6. \end{cases}$$

Thus the signs of the values $\phi(C_{n,g};1)$ are positive when $g \equiv 1,2,5 \pmod 6$ and negative when $g \equiv 3,4 \pmod 6$. Hence $\phi(C_{n,g};\alpha)$ and $\phi(C_{n,g};1)$ have opposite signs. □

Remark 3.4. In Proposition 3.3, the lower bound is improved by 1. We can rephrase Propositions 3.2 and 3.3 as follows:

If $g \not\equiv 0 \pmod 6$, then

$$m_{C_{n,g}}[0,1) \geq \left\lceil \frac{d(C_{n,g})+1}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1.$$

If $g \equiv 0 \pmod 6$, then

$$m_{C_{n,g}}[0,1) \geq \left\lceil \frac{d(C_{n,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1.$$

Lemma 3.5. *Let C_n be the cycle graph with n vertices and let $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ and $y = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^T$ be vectors of order n, where*

$$x_i = \begin{cases} 1, & \text{if } i \equiv 1,\ 6 \pmod 6, \\ 0, & \text{if } i \equiv 2,\ 5 \pmod 6, \\ -1, & \text{if } i \equiv 3,\ 4 \pmod 6 \end{cases} \quad \text{and} \quad y_i = \begin{cases} 1, & \text{if } i \equiv 1,\ 2 \pmod 6, \\ 0, & \text{if } i \equiv 3,\ 6 \pmod 6, \\ -1, & \text{if } i \equiv 4,\ 5 \pmod 6. \end{cases}$$

If n is divisible by 6, then the vectors x and y are eigenvectors corresponding to the Laplacian eigenvalue 1 of C_n.

Proof. It can be verified by direct computation. □

The following proposition provides the conditions under which a lollipop graph has 1 as a Laplacian eigenvalue.

Proposition 3.6. *Let $C_{n,g}$ be a lollipop graph with n vertices and girth g. Let x be an eigenvector corresponding to the Laplacian eigenvalue 1 of P_n in Lemma 2.9 and let y be an eigenvector corresponding to the Laplacian eigenvalue 1 of C_g in Lemma 3.5. Then*

(a) *If $g \equiv 0 \pmod 6$, then 1 is a Laplacian eigenvalue of $C_{n,g}$ with the corresponding eigenvector z, where the vector z has coordinates*

$$z_i = \begin{cases} y_i, & \text{if } i = 1, \ldots, g, \\ 0, & \text{if } i = g+1, \ldots, n. \end{cases}$$

Moreover,

$$m_{C_{n,g}}(1) = \begin{cases} 2, & \text{if } n \equiv 0 \pmod 3, \\ 1, & \text{if } n \not\equiv 0 \pmod 3. \end{cases}$$

(b) *If $g \equiv 1 \pmod 6$ and $n \equiv 0 \pmod 3$, then 1 is a Laplacian eigenvalue of $C_{n,g}$ with the corresponding eigenvector x and $m_{C_{n,g}}(1) = 1$.*

(c) *If $g \equiv 3 \pmod 6$ and $n \equiv 1 \pmod 3$, then 1 is a Laplacian eigenvalue of $C_{n,g}$ with the corresponding eigenvector w, where the vector w has coordinates*

$$w_i = \begin{cases} y_i, & \text{if } i = 1, \ldots, g, \\ -2y_{i-g}, & \text{if } i = g+1, \ldots, n. \end{cases}$$

Moreover, $m_{C_{n,g}}(1) = 1$.

Proof. (a) Suppose that $g \equiv 0 \pmod 6$. Note that $L(C_{n,g}) - L(C_r \cup P_{n-g}) = M = (m_{ij})$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \{(g,g),\ (g+1,g+1)\}, \\ -1, & \text{if } (i,j) \in \{(g,g+1),\ (g+1,g)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $g \equiv 0 \pmod 6$, the coordinate z_g of the vector z is 0 and $L(C_g)y = y$. Since $z_g = z_{g+1} = 0$, we obtain $Mz = 0$. Then

$$L(C_{n,g})z = L(C_g \cup P_{n-g})z + Mz = z.$$

Thus 1 is a Laplacian eigenvalue with the corresponding eigenvector z.

Suppose that $n \equiv 0 \pmod 3$ and $g \equiv 0 \pmod 6$. Then $n - g \equiv 0 \pmod 3$. Thus, by Corollary 2.7, $m_{C_{n,g}}(1) = m_{C_g}(1) = 2$. Now, we claim that $m_{C_{n,g}}(1) = 1$ if $g \equiv 0 \pmod 6$ and $n \not\equiv 0 \pmod 3$. Suppose that $m = m_{C_{n,g}}(1) > 1$. Then

$$\mu_k(C_{n,g}) = \cdots = \mu_{k+m-1}(C_{n,g}) = 1$$

for some k. Since P_n is a subgraph of $C_{n,g}$, by Lemma 2.1,

$$1 = \mu_k(C_{n,g}) \le \mu_{k+1}(P_n) \le \mu_{k+1}(C_{n,g}) = 1.$$

Thus $\mu_{k+1}(P_n) = 1$, which contradicts to $m_{P_n}(1) = 0$. Hence $m_{C_{n,g}}(1) = 1$.

(b) Suppose that $g \equiv 1 \pmod 6$ and $n \equiv 0 \pmod 3$. Note that $L(C_{n,g}) - L(P_n) = N = (n_{ij})$, where

$$n_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \{(1,1),\ (g,g)\}, \\ -1, & \text{if } (i,j) \in \{(1,g),\ (g,1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $g \equiv 1 \pmod 6$, we have $x_g = 1$. Since $x_1 = x_g = 1$, we obtain $Nx = 0$. Since $n \equiv 0 \pmod 3$, we have $L(P_n)x = x$. Then

$$L(C_{n,g})x = L(P_n)x + Nx = x.$$

Thus 1 is a Laplacian eigenvalue with the corresponding eigenvector x.

Now, we show that $m_{C_{n,g}}(1) = 1$. We consider the subgraph $C_g \cup P_{n-g}$ of $C_{n,g}$. Since $g \equiv 1 \pmod 6$ and $n \equiv 0 \pmod 3$, we have $n - g \not\equiv 0 \pmod 3$. Thus neither C_g nor P_{n-g} have the Laplacian eigenvalue 1. Suppose that $m = m_{C_{n,g}}(1) > 1$. Assume that

$$\mu_k(C_{n,g}) = \cdots = \mu_{k+m-1}(C_{n,g}) = 1$$

for some k. Since $C_g \cup P_{n-g}$ is a subgraph of $C_{n,g}$, by Lemma 2.1,

$$1 = \mu_k(C_{n,g}) \le \mu_{k+1}(C_g \cup P_{n-g}) \le \mu_{k+1}(C_{n,g}) = 1.$$

Thus $\mu_{k+1}(C_g \cup P_{n-g}) = 1$, which contradicts to $m_{C_g \cup P_{n-g}}(1) = 0$. Hence $m_{C_{n,g}}(1) = 1$.

(c) Let l_{ij} be the (i,j)-entry of $L(C_{n,g})$. To prove $L(C_{n,g})w = w$, we compute the coordinates of $L(C_{n,g})w$. For $i \notin \{1, g, n\}$, the entries of $L(C_{n,g})$ are

$$l_{ij} = \begin{cases} 2, & \text{if } j = i, \\ -1, & \text{if } j = i-1, i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\begin{bmatrix} w_{i-1} & w_i & w_{i+1} \end{bmatrix}^T$ is one of

$$c\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \; c\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T, \; \text{or } c\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T,$$

where $c = \pm 1, \pm 2$, we obtain

$$\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} w_{i-1} & w_i & w_{i+1} \end{bmatrix}^T = w_i.$$

Hence the ith coordinate of $L(C_{n,g})w$ is equal to w_i for $i \notin \{1, g, n\}$. Now, we examine the remaining cases $i \in \{1, g, n\}$. Since $w_1 = w_2 = 1$, $w_g = 0$, and the entry of the first row of $L(C_{n,g})$ is

$$l_{1j} = \begin{cases} 2, & \text{if } j = 1, \\ -1, & \text{if } j = 2 \text{ and } g, \\ 0, & \text{otherwise,} \end{cases}$$

the first coordinate of $L(C_{n,g})w$ is $w_1 = 1$. Since $w_1 = w_{g-1} = 1$, $w_r = 0$, $w_{g+1} = -2$ and the entry of the gth row of $L(C_{n,g})$ is

$$l_{gj} = \begin{cases} 3, & \text{if } j = g, \\ -1, & \text{if } j = 1,\ g-1 \text{ and } g+1, \\ 0, & \text{otherwise,} \end{cases}$$

the gth coordinate of $L(C_{n,g})w$ is $w_g = 0$. Since $w_{n-1} = 0$, $w_n = \pm 2$ and the entry of the last row of $L(C_{n,g})$ is

$$l_{nj} = \begin{cases} -1, & \text{if } j = n-1, \\ 1, & \text{if } j = n, \\ 0, & \text{otherwise,} \end{cases}$$

the last coordinate of $L(C_{n,g})w$ is $w_n = \pm 2$. Thus $L(C_{n,g})w = w$.

Now, we show that $m_{C_{n,g}}(1) = 1$. Suppose that $m = m_{C_{n,g}}(1) > 1$. Then

$$\mu_k(C_{n,g}) = \cdots = \mu_{k+m-1}(C_{n,g}) = 1$$

for some k. Since P_n is a subgraph of $C_{n,g}$, by Lemma 2.1,

$$1 = \mu_k(C_{n,g}) \le \mu_{k+1}(P_n) \le \mu_{k+1}(C_{n,g}) = 1.$$

Thus $\mu_{k+1}(P_n) = 1$, which contradicts to $m_{P_n}(1) = 0$. Hence $m_{C_{n,g}}(1) = 1$. □

3.2. Compass graphs

Finally, we determine the lower bound for $m_G[0,1)$, where G is a compass graph in Fig. 1. We label the n vertices of the graph as shown in Fig. 2.

Proposition 3.7. *Let $C_{n,g}(g',t)$ be a graph with n vertices and girth g. Suppose that $n \equiv t + \lceil \frac{g}{2} \rceil \pmod 3$ or $t \equiv 0 \pmod 3$. Then*

$$m_{C_{n,g}(g',t)}[0,1) \ge \left\lceil \frac{d(C_{n,g}(g',t))}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1.$$

Proof. We set $G = C_{n,g}(g', t)$. Note that $s = n - g - t$ and $g' = d(G) - t - s$. Let $k = \lceil \frac{d(G)}{3} \rceil + \left\lceil \frac{g}{6} \right\rceil - 1$. A subgraph $P_t \cup C_{n-t,g}$ can be obtained from G by deleting the edge $v_t v_{t+g'}$, where v_t and $v_{t+g'}$ are labeled as in Fig. 2 (a). If $g' = 0$, then subgraph $P_t \cup C_{n-t,g}$ can be obtained from G by deleting the edge $v_t v_{t+g}$, where v_t and v_{t+g} are labeled as in Fig. 2 (b). Let $\alpha = \lfloor \frac{g}{2} \rfloor - g'$. Since $\lfloor \frac{g}{2} \rfloor \geq g'$, we have $\alpha \geq 0$. Note that $d(C_{n-t,g}) + t = d(G) + \alpha$.

Case 1. Suppose that $t \equiv 0 \pmod 3$. By Corollary 2.7, we have $m_G(1) = m_{C_{n-t,g}}(1)$. Put $l = m_{C_{n-t,g}}(1)$. Then $m_{P_t \cup C_{n-t,g}}(1) = l + 1$. Note that

$$\begin{aligned} m_{P_t \cup C_{n-t,g}}[0,1) &\geq \left\lceil \frac{d(C_{n-t,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 + \left\lceil \frac{t}{3} \right\rceil \\ &= \left\lceil \frac{d(G)+\alpha}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 \\ &\geq \left\lceil \frac{d(G)}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 = k. \end{aligned}$$

By Lemma 2.1, we obtain $\mu_{k-1}(G) \leq \mu_k(P_t \cup C_{n-t,g}) < 1$. Suppose that $\mu_k(G) \geq 1$. By Lemma 2.1, we have

$$1 \leq \mu_k(G) \leq \mu_{k+1}(P_t \cup C_{n-t,g}) \leq \mu_{k+1}(G) \leq \cdots \leq \mu_{k+l}(G) \leq \mu_{k+l+1}(P_t \cup C_{n-t,g}) = 1.$$

It follows that

$$\mu_k(G) = \mu_{k+1}(G) = \cdots = \mu_{k+l}(G) = 1.$$

Thus, we obtain $m_G(1) = l + 1$. This contradicts to $m_G(1) = l$. Thus, $\mu_k(G) < 1$ and hence $m_G[0,1) \geq k$.

Case 2. Suppose that $d(C_{n-t,g}) = n - t - \lceil \frac{g}{2} \rceil \equiv 0 \pmod 3$. If $g \not\equiv 0 \pmod 6$, by Proposition 3.3, we have

$$\begin{aligned} m_{P_t \cup C_{n-t,g}}[0,1) &= \left\lceil \frac{d(C_{n-t,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil + \left\lceil \frac{t}{3} \right\rceil \\ &= \left\lceil \frac{d(G)+\alpha}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil \\ &\geq \left\lceil \frac{d(G)}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil = k+1. \end{aligned}$$

Thus $\mu_k(G) \leq \mu_{k+1}(P_t \cup C_{n-t,g}) < 1$ and hence $m_G[0,1) \geq k$. If $g \equiv 0 \pmod 6$, then $0 \equiv d(C_{n-t,g}) = \lfloor \frac{g}{2} \rfloor + s \equiv s \pmod 3$. In this case, we consider the subgraph $P_s \cup C_{n-s,g}$. Since $s \equiv 0 \pmod 3$, by Case 1, we obtain $m_G[0,1) \geq k$. □

Lemma 3.8. *Let $C_{n,g}(g', t)$ be a graph with n vertices and girth g. Suppose that $g \equiv 0 \pmod 6$ and $g' = \frac{g}{2}$. Then $C_{n,g}(g', t)$ has a Laplacian eigenvalue 1.*

Proof. Let $G = C_{n,g}(g', t)$ and let $P_t \cup C_{n-t,g}$ be a subgraph of G obtained by deleting the edge $v_t v_{t+g'}$, where v_t and $v_{t+g'}$ are labeled as in Fig. 2. Note that $L(G) - L(P_t \cup C_{n-t,g}) = A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \{(t,t),\ (t+g', t+g')\}, \\ -1, & \text{if } (i,j) \in \{(t, t+g'),\ (t+g', t)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let y be the eigenvector corresponding to the Laplacian eigenvalue 1 of C_r in Lemma 3.5 and let z be a vector of order n with coordinates

$$z_i = \begin{cases} y_{i-t}, & \text{if } i = t+1, \ldots, t+g, \\ 0, & \text{otherwise.} \end{cases}$$

Since $z_t = 0$ and $z_{t+g'} = y_{g'} = 0$, we have $Az = \mathbf{0}$. Then, by Proposition 3.6 (a),

$$L(G)z = L(P_t \cup C_{n-t,g})z + Az = z.$$

Thus G has a Laplacian eigenvalue 1. □

A lower bound for the compass graph is provided under specific conditions in the following.

Proposition 3.9. *Let $C_{n,g}(g', t)$ be a graph with n vertices and girth g. Suppose that $n \equiv 0 \pmod 3$, $g \equiv 0 \pmod 6$, $g' = \frac{g}{2}$, and $t \equiv 1 \pmod 3$. Then*

$$m_{C_{n,g}(g',t)}[0,1) \geq \left\lceil \frac{d(C_{n,g}(g',t))}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil.$$

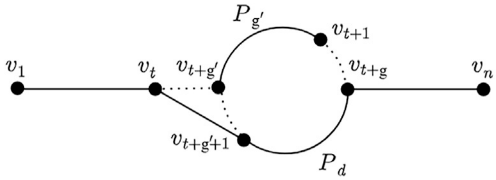

Fig. 7. The graph $P_d \cup P_{g'}$ obtained from $C_{n,g}(g',t)$, where $d = d(C_{n,g}(g',t))$.

Proof. To simplify the notation, put $G = C_{n,g}(g',t)$ and $d = d(C_{n,g}(g',t))$. Let $k = \lceil \frac{d}{3} \rceil + \lceil \frac{g}{6} \rceil$. First, we consider a subgraph $P_t \cup C_{n-t,g}$ obtained from G by removing the edge $v_t v_{t+g'}$. Note that $t \equiv 1 \pmod 3$ and $d(C_{n-t,g}) = n - t - \lceil \frac{g}{2} \rceil \equiv 2 \pmod 3$. Then

$$m_{P_t \cup C_{n-t,g}}[0,1) \geq \left\lceil \frac{d(C_{n-t,g})}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1 + \left\lceil \frac{t}{3} \right\rceil = \left\lceil \frac{d(G)}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil = k.$$

By Lemma 2.1, we have $\mu_{k-1}(G) \leq \mu_k(P_t \cup C_{n-t,g}) < 1$ and hence $m_G[0,1) \geq k-1$. Suppose that $\mu_k(G) \geq 1$. Since the graph $P_t \cup C_{n-t,g}$ has the Laplacian eigenvalue 1, by Lemma 3.8, we obtain $\mu_k(G) = 1$. Now, we consider the graph $P_d \cup P_{g'}$ which is obtained from G by deleting three edges $v_t v_{t+g'}$, $v_{t+1} v_{t+g}$, and $v_{t+g'} v_{t+g'+1}$ and adding an edge $v_t v_{t+g'+1}$ (see Fig. 7). Since $d \equiv 0 \pmod 3$ and $g' \equiv 0 \pmod 3$, we have

$$\mu_{k+1}(P_d \cup P_{g'}) = \mu_{k+2}(P_d \cup P_{g'}) = 1.$$

Note that $L(G) - L(P_d \cup P_{g'}) = B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 2, & \text{if } (i,j) \in \{(t+g', t+g')\}, \\ 1, & \text{if } (i,j) \in \{(t, t+g'+1), (t+1, t+1), (t+g'+1, t), (t+g, t+g)\}, \\ -1, & \text{if } (i,j) \in \{(t, t+g'), (t+1, t+g), (t+g', t), (t+g', t+g'+1), \\ & \qquad (t+g'+1, t+g'), (t+g, t+1)\}, \\ 0, & \text{otherwise}. \end{cases}$$

Since B is permutationally similar to

$$\begin{bmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \oplus O_{n-5},$$

where O_{n-5} is the square zero matrix of order $n-5$, we have

$$\mu_1(B) = -1, \ \mu_2(B) = \cdots = \mu_{n-2}(B) = 0, \ \mu_{n-1}(B) = 2, \ \text{and} \ \mu_n(B) = 3.$$

Thus $\mu_k(G) = \mu_{k+2}(P_d \cup P_{g'}) + \mu_{n-2}(B)$. By Lemma 2.5, there exists an eigenvector z such that

$$L(G)z = \mu_k(G)z = z,$$

$$L(P_d \cup P_{g'})z = \mu_{k+2}(P_d \cup P_{g'})z = z, \ \text{and}$$

$$Bz = \mu_{n-2}(B)z = \mathbf{0}.$$

Now, we show that there is no common eigenvector satisfying the above three equations. Let x and x' be the eigenvector corresponding to the Laplacian eigenvalue 1 of P_d and $P_{g'}$, respectively. We define two vectors z and z' of order n such that

$$z_i = \begin{cases} x_i, & \text{if } i \in \{1, \ldots, t\}, \\ 0, & \text{if } i \in \{t+1, \ldots, t+g'\}, \\ x_{i-g'}, & \text{if } i \in \{t+g'+1, \ldots, n\} \end{cases} \quad \text{and} \quad z'_i = \begin{cases} x'_{i-t}, & \text{if } i \in \{t+1, \ldots, t+g'\}, \\ 0, & \text{otherwise}. \end{cases}$$

Then z and z' are eigenvectors corresponding to the Laplacian eigenvalue 1 of $P_d \cup P_{g'}$. First, we show that $Bz \neq \mathbf{0}$. Note that

$$z_t = x_t, \ z_{t+1} = z_{t+g'} = 0, \ z_{t+g'+1} = x_{t+1}, \ \text{and} \ z_{t+g} = x_{t+g'}.$$

Suppose that $g = 6l$ for some odd integer l. Then $g' \equiv 3 \pmod 6$. If $t \equiv 1 \pmod 6$, then

$$z_t = 1, \ z_{t+g'+1} = 0 \ \text{and} \ z_{t+g} = -1.$$

If $t \equiv 4 \pmod 6$, then

$$z_t=-1,\ z_{t+g'+1}=0 \ \text{ and } \ z_{t+g}=1.$$

Suppose that $g=6l'$ for some even integer l'. Then $g'\equiv 0 \pmod 6$. If $t\equiv 1 \pmod 6$, then

$$z_t=1,\ z_{t+g'+1}=0 \ \text{ and } \ z_{t+g}=1.$$

If $t\equiv 4 \pmod 6$, then

$$z_t=-1,\ z_{t+g'+1}=0 \ \text{ and } \ z_{t+g}=-1.$$

In each case, we obtain $Bz\neq\mathbf{0}$. Now, we prove that $Bz'\neq\mathbf{0}$. Note that

$$z'_t=0,\ z'_{t+1}=x'_1=1,\ z'_{t+g'}=x'_{g'},\ \text{and } z'_{t+g'+1}=z'_{t+g}=0.$$

Since

$$z'_{t+g'}=\begin{cases} 1 & \text{if } g'\equiv 0 \pmod 6,\\ -1 & \text{if } g'\equiv 3 \pmod 6,\end{cases}$$

we have $Bz'\neq\mathbf{0}$. This is a contradiction to $\mu_k(G)\geq 1$. Hence $m_G[0,1)\geq k$. □

Proposition 3.10. *Let $C_{n,g}(g',t)$ be a graph with n vertices and girth g. Suppose that $n\not\equiv t+\lceil\frac{g}{2}\rceil \pmod 3$ and $t\not\equiv 0 \pmod 3$. Then*

$$m_{C_{n,g}(g',t)}[0,1)\geq\left\lceil\frac{d(C_{n,g}(g',t))}{3}\right\rceil+\left\lceil\frac{g}{6}\right\rceil-1.$$

Proof. We set $G=C_{n,g}(g',t)$. Note that $s=n-g-t$ and $g'=d(G)-t-s$. Let $k=\lceil\frac{d(G)}{3}\rceil+\left\lceil\frac{g}{6}\right\rceil-1$. A subgraph $P_t\cup C_{n-t,g}$ can be obtained from G by deleting the edge $v_tv_{t+g'}$ if $g'\neq 0$ or the edge v_tv_{t+g} if $g'=0$. Then $d(C_{n-t,g})=n-t-\lceil\frac{g}{2}\rceil$. Let $\alpha=\lfloor\frac{g}{2}\rfloor-g'$. Since $\lfloor\frac{g}{2}\rfloor\geq g'$, we have $\alpha\geq 0$. Note that $d(C_{n-t,g})+t=d(G)+\alpha$. We consider the following two cases:

(1) $d(C_{n-t,g})\not\equiv 2 \pmod 3$ or $t\not\equiv 2 \pmod 3$.
(2) $d(C_{n-t,g})\equiv 2 \pmod 3$ and $t\equiv 2 \pmod 3$.

Case 1. Suppose that $d(C_{n-t,g})\not\equiv 2 \pmod 3$ or $t\not\equiv 2 \pmod 3$. Then

$$\begin{aligned} m_{C_{n-t,g}\cup P_t}[0,1) &\geq \left\lceil\frac{d(C_{n-t,g})}{3}\right\rceil+\left\lceil\frac{g}{6}\right\rceil-1+\left\lceil\frac{t}{3}\right\rceil\\ &> \left\lceil\frac{d(G)+\alpha}{3}\right\rceil+\left\lceil\frac{g}{6}\right\rceil-1\\ &\geq \left\lceil\frac{d(G)}{3}\right\rceil+\left\lceil\frac{g}{6}\right\rceil-1=k. \end{aligned}$$

Thus, by Lemma 2.1, we obtain $\mu_k(G)\leq\mu_{k+1}(P_t\cup C_{n-t,g})<1$ and hence $m_G[0,1)\geq k$.

Case 2. Suppose that $d(C_{n-t,g})\equiv 2 \pmod 3$ and $t\equiv 2 \pmod 3$. If $s\not\equiv 2 \pmod 3$, then we consider the subgraph $P_s\cup C_{n-s,g}$. Thus, by Case 1, we have $m_G[0,1)\geq k$. Assume that $s\equiv 2 \pmod 3$. For $\alpha\geq 3$, we have

$$m_{P_t\cup C_{n-t,g}}[0,1)\geq\left\lceil\frac{d(G)+\alpha}{3}\right\rceil+\left\lceil\frac{g}{6}\right\rceil-1\geq\left\lceil\frac{d(G)}{3}\right\rceil+\left\lceil\frac{g}{6}\right\rceil=k+1.$$

By Lemma 2.1, $\mu_k(G)\leq\mu_{k+1}(P_t\cup C_{n-t,g})<1$. Thus $m_G[0,1)\geq k$ for $\alpha\geq 3$. Now, we assume that $0\leq\alpha<3$.

Case 2-1. Suppose that $d(G)\not\equiv 1 \pmod 3$. Let G' be the graph obtained from G by deleting the vertex v_1 in Fig. 2. Then, by Case 1, we have

$$m_{G'}[0,1)\geq\left\lceil\frac{d(G)-1}{3}\right\rceil+\left\lceil\frac{g}{6}\right\rceil-1=\left\lceil\frac{d(G)}{3}\right\rceil+\left\lceil\frac{g}{6}\right\rceil-1.$$

Thus, by Lemma 2.2, we obtain

$$m_G[0,1)\geq\left\lceil\frac{d(G)}{3}\right\rceil+\left\lceil\frac{g}{6}\right\rceil-1.$$

Case 2-2. Suppose that $d(G)\equiv 1 \pmod 3$. Since $d(C_{n-t,g})=\lfloor\frac{g}{2}\rfloor+s\equiv 2 \pmod 3$, we have $\lfloor\frac{g}{2}\rfloor\equiv 0 \pmod 3$, that is, $g\equiv 0,1 \pmod 6$. Since

$$d(G)=g'+s+t\equiv 1 \pmod 3,\quad t\equiv 2 \pmod 3,\quad\text{and}\quad s\equiv 2 \pmod 3,$$

we obtain $g' \equiv 0 \pmod 3$. If $g' = 0$, then $\alpha = \lfloor \frac{g}{2} \rfloor \equiv 0 \pmod 3$. Since $0 \le \alpha < 3$, we have $\lfloor \frac{g}{2} \rfloor = 0$, that is, $g = 0, 1$, which is a contradiction. Suppose that $g' \neq 0$. Then $\alpha = \lfloor \frac{g}{2} \rfloor - g'$ must be zero. Hence the graph G satisfies the following conditions:

$$g \equiv 0, 1 \pmod 6, \; g' = \left\lfloor \frac{g}{2} \right\rfloor, \text{ and } t \equiv s \equiv 2 \pmod 3.$$

Suppose that $g \equiv 0 \pmod 6$. Since the subgraph G' is the compass graph $C_{n-1,g}(g', t-1)$, by Proposition 3.9, we have

$$m_{G'}[0,1) \ge \left\lceil \frac{d(G)-1}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil = \left\lceil \frac{d(G)}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1.$$

Hence $m_G[0,1) \ge k$, by Lemma 2.2.

Suppose that $g \equiv 1 \pmod 6$. Since $m_{P_t \cup C_{n-t,g}}[0,1) \ge k$, we have

$$m_G[0,1) \ge k-1.$$

Suppose that $\mu_k(G) \ge 1$. Since $n = g + t + s \equiv 0 \pmod 3$, by Proposition 3.6 (b), the lollipop $C_{n-t,g}$ has the Laplacian eigenvalue 1 with multiplicity 1. Thus, by Lemma 2.1, we have $\mu_k(G) = \mu_{k+1}(P_t \cup C_{n-t,g}) = 1$. Note that $L(G) - L(P_t \cup C_{n-t,g}) = A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \{(t,t),\ (t+g', t+g')\}, \\ -1, & \text{if } (i,j) \in \{(t, t+g'),\ (t+g', t)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since A is permutationally similar to $L(K_2 \cup (n-2)K_1)$, we have

$$\mu_1(A) = \cdots = \mu_{n-1}(A) = 0 \text{ and } \mu_n(A) = 2.$$

Then $\mu_k(G) = \mu_{k+1}(P_t \cup C_{n-t,g}) + \mu_{n-1}(A)$. By Lemma 2.5, there exists an eigenvector z such that

$$L(G)z = \mu_k(G)z = z,$$
$$L(P_t \cup C_{n-t,g})z = \mu_{k+1}(P_t \cup C_{n-t,g})z = z, \text{ and}$$
$$Az = \mu_{n-1}(A)z = \mathbf{0}.$$

Then z must be a vector of the form

$$z = \begin{bmatrix} \mathbf{0} \\ \hline x \end{bmatrix},$$

where the first t coordinates are 0 and x is the eigenvector corresponding to the Laplacian eigenvalue 1 of $C_{n-t,g}$. By Proposition 3.6 (b), the coordinate $z_{t+g'}$ of the vector z is -1. Since $z_t = 0$ and $z_{t+g'} = -1$, we obtain $Az \neq \mathbf{0}$. This is a contradiction. Thus, $\mu_k(G) < 1$ and hence $m_G[0,1) \ge k$. □

4. Conclusion

Let G be unicyclic with diameter $d(G)$ and girth g. Let G' be a minimal unicyclic graph of G that contains a diametral path. Then G' is one of the cycle, lollipop, or compass graphs with diameter $d(G') = d(G)$ and girth g. By combining results in Section 3, we have

$$m_{G'}[0,1) \ge \left\lceil \frac{d(G')}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1.$$

By Lemma 2.2, we obtain

$$m_G[0,1) \ge \left\lceil \frac{d(G)}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1.$$

Thus we have the following theorem.

Theorem 4.1. *Let G be a unicyclic graph with diameter $d(G)$ and girth g. Then*

$$m_G[0,1) \ge \left\lceil \frac{d(G)}{3} \right\rceil + \left\lceil \frac{g}{6} \right\rceil - 1.$$

Data availability

No data was used for the research described in the article.

Acknowledgements

Sunyo Moon was supported by the KIAS Individual Grant CG092301 at Korea Institute for Advanced Study. The second author wishes to thank the Korea Institute for Advanced Study for its generous hospitality and support during the time this paper was written.

References

[1] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Monographs and Textbooks in Pure and Applied Mathematics, vol. 208, Marcel Dekker, Inc., New York, 1998.
[2] C.X. Kang, On domination number and distance in graphs, Discrete Appl. Math. 200 (2016) 203–206.
[3] S.T. Hedetniemi, D.P. Jacobs, V. Trevisan, Domination number and Laplacian eigenvalue distribution, Eur. J. Comb. 53 (2016) 66–71.
[4] J. Guo, J. Xue, R. Liu, Laplacian eigenvalue distribution, diameter and domination number of trees, in: Linear Multilinear Algebra, 2024.
[5] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (1990) 218–238.
[6] R.A. Horn, C.R. Johnson, Matrix Analysis, second ed., Cambridge University Press, Cambridge, 2013.
[7] J.-M. Guo, On the second largest Laplacian eigenvalue of trees, Linear Algebra Appl. 404 (2005) 251–261.
[8] J.-M. Guo, A conjecture on the algebraic connectivity of connected graphs with fixed girth, Discrete Math. 308 (2008) 5702–5711.